Differential Equations

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1 Introduction

A *differential equation* is an equation relating one or more individual variables, one or more dependent variables, and their derivatives.

An *ordinary differential equation (ODE)* is a differential equation that only involves one independent variable, the rest being functions of that variable. For example,

$$y' = x + 2$$

is an example of an ODE.

A *partial differential equation (PDE)* is a differential equation that involves more than one independent variables, with multivariate functions. The Schrodinger equation is an example of this:

$$\frac{-\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} = i\hbar\frac{\partial \Psi}{\partial t}$$

The *order* of a differential equation defined by the highest derivative that appears in it. For example,

$$y'' = y' + x$$

is a second-order differential equation.

An *initial value problem (IVP)* is a diffferential equation with initial conditions for each of its derivatives at a single point. For an nth order differential equation, initial conditions may take the form

$$\begin{aligned} y(a) &= y_0 \\ y'(a) &= y_1 \\ y''(a) &= y_2 \\ &\vdots \\ y^{(n-1)} &= y_{n-1} \end{aligned}$$

A *boundary value problem (BVP)* provides information about the function at multiple points. The distinction between an IVP and a BVP is that BVPs provide information from multiple points.

In general, a *solution* to a differential equation is a function that satisfies that differential equation.

An *explicit* solution is a non-trivial function that satisfies the differential equation on some $I \subseteq \mathbb{R}$. This means that the solution is of the form

$$y = f(x)$$

An *implicit* solution is a relation between variables that fulfills the differential equation. This relation can take the form

$$f(x, y) = g(x, y)$$

A *particular* solution is a solution to a differential equation with no unknown constants. For an IVP, finding a particular solution means that initial values have been substituted in and constants have been solved for. In contrast, a *general* solution leaves unknown constants.

A *linear* differential equation is one of the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x) = 0$$

where each a_i is a function of x. In other words, it is a linear combination of the derivatives of why.

A *homogeneous* differential equation is where $a(x) = 0$, so that every term in the equation involves some derivative of y or y itself.

2 First Order Differential Equations

2.1 Integration

Many differential equations are solved by first manipulating one side into resembling

$$\frac{dy}{dx} = f(x)$$

and then taking an integral on both sides. This yields some constant C, which is solved for in IVPs.

2.1.1 Existence & Uniqueness #1

Given a differential equation of the form

$$y' = f(x, y)$$

with some initial condition

$$y(a) = b$$

If f is continuous on some region containing (a, b), then we are guaranteed the existence of at least one particular solution to the differential equation on that region.

If f_y is continuous on that same region containing (a, b), then we are guaranteed that the particular solution is unique.

2.2 Separable Equations

Some differential equations are *separable*, meaning that an equation of the form

$$y' = f(x, y)$$

can be manipulated into something of the form

$$y' = g(x) \cdot h(y)$$

We may then "separate" the equation, yielding

$$\begin{aligned} \frac{1}{h(y)} \frac{dy}{dx} &= g(x) \\ \int \frac{1}{h(y)} \frac{dy}{dx} dx &= \int g(x) dx \\ \int \frac{1}{h(y)} dy &= \int g(x) dx \end{aligned}$$

The last equation is something that can be solved for using single-variable calculus techniques.

2.3 First Order Linear Differential Equations

Given a first order linear differential equation of a monic form

$$y' + P(x) \cdot y = Q(x)$$

We may attempt to manipulate the left hand side into a product rule, which would just involve the dependent variable y itself with some function of x. To do this, we can choose some function ρ such that

$$\rho(x) = e^{\int P(x) dx}$$

To see why, we can take the derivative of $\rho(x)$, which yields

$$\rho'(x) = \frac{d}{dx} \left(e^{\int P(x) dx} \right) = e^{\int P(x) dx} \cdot P(x)$$

by the chain rule. If we multiply the entire differential equation by ρ,

$$\rho \cdot y' + \rho \cdot P(x) \cdot y = \rho \cdot Q(x)$$

The left side can be condensed into a derivative:

$$\frac{d}{dx}(\rho \cdot y) = \rho \cdot Q(x)$$

By integrating,

$$\rho \cdot y = \int \rho \cdot Q(x) dx$$

which will yield a solution to our differential equation.

Note that, in order to apply this method, the differential equation must be *monic*; in other words, the coefficient of y' must be 1.

2.3.1 Existence & Uniqueness # 2

For a differential equation

$$y' = Q(x) - P(x)y, \ y(x_0) = y_0$$

if Q and P are continuous on some interval $I \subseteq \mathbb{R}$, where $x_0 \in I$, then there exists a unique solution $y(x)$ on I.

2.4 Substitution

For a differential equation of the form

$$y' = f(x, y)$$

which is neither necessarily separable or linear, we may find some function

$$v = \alpha(x, y)$$

that we may substitute into the equation. The function can be solved for in terms of y, yielding some other function

$$y = \mathbf{\Phi}(x, v), \ (v = v(x))$$

where applying the multivariable chain rule would yield

$$y' = \frac{\partial \mathbf{\Phi}}{\partial x}\frac{dx}{dx} + \frac{\partial \mathbf{\Phi}}{\partial v}\frac{dv}{dx}$$

$$y' = \mathbf{\Phi}_x + \mathbf{\Phi}_v \cdot v' = f(x, \mathbf{\Phi}(x, v))$$

which can be rearranged into

$$v' = \frac{\mathbf{\Phi}(x, v) - \mathbf{\Phi}_x}{\mathbf{\Phi}_v} = g(x, v)$$

This resulting differential equaiton can either be separable or linear, which will allow us to solve in terms of the substitution v. After we have done so, we may simply substitute the original α back into the equation.

There are several common substitution sused for equations of specific types.

1. $v = \frac{y}{x}$ for $y' = F\left(\frac{y}{x}\right)$
2. $v = ax + by + c$ for $y' = F(ax + by + c)$
3. $v = y^{1-n}$ for $y' + P(x)y = Q(x)y^n$, where $n \neq 0, 1$
4. $v = \ln y$ for $y' + P(x)y = Q(x)y \ln y$

In each of these cases, the substitution is guaranteed to produce an equation that is either separable or linear.

2.5 Exact Equations

The general solution of a first order differential equation can be expressed by an equation of the form

$$F(x, y(x)) = C$$

Through the chain rule, this becomes

$$F_x \cdot \frac{dx}{dx} + F_y \cdot \frac{dy}{dx} = 0$$

$$F_x dx + F_y dy = 0$$

The above is known as the differential form. The general case of a differential form is the equation

$$M(x, y)dx + N(x, y)dy = 0$$

For an equation F, if $M = F_x$ and $N = F_y$, then the differential form is "exact". Even further,

$$F(x, y) = C$$

is a solution to the differential equation. Finding this exact equation is analogous to determining whether or not a vector field is conservative in multivariable calculus.

While some differential forms have corresponding F functions, many do not. However, they can be made exact by multiplying by some function ρ, which will take one of the forms

$$\rho = e^{\int \frac{N_x - M_y}{M} dy}$$
$$\rho = e^{\int \frac{M_y - N_x}{N} dx}$$

2.6 Applications

2.6.1 Tank Problems

For some tank with volume V and an amount of particulate of matter y, we may model the change in matter inside of the tank with a differential equation. Intuitively, the change in matter is the difference between the rate at which matter enters the tank and the rate at which matter leaves the tank.

$$\frac{dy}{dt} = \text{matter in} - \text{matter out}$$

By substituting in known quantities, we should end up with either a linear or separable differential equation with which we may find the function $y(x)$ that describes the concentration of matter in the tank at some time t.

3 nth Order Linear Differential Equations

3.1 Linear Theory

A second order linear differential equation is something of the form

$$G(x, y, y', y'') = 0$$

where G outputs a linear combination of its four inputs. In other words,

$$A(x)y'' + B(x)y' + C(x)y = F(x)$$

This differential equation becomes homogeneous if there is no $F(x)$ term; in other words, if $F(x) = 0$. If a differential equation is not homogeneous, its associate homogeneous equation is the linear combination of derivatives set to 0.

In the general case, a linear differential equation will look something like

$$a_n(x)y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0(x)y = f(x)$$

We introduce the L operator, which is just a convenient notation for the above equation.

$$L(y) = a_n(x)y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0(x)y$$

If we input some function $y_1 + y_2$ into L, with some rearrangement, we can prove that

$$L(y_1 + y_2) = L(y_1) + L(y_2)$$

Furthermore,

$$L(cy_1) = cL(y_1)$$

The L operator is, therefore, a linear function.

3.1.1 Existence & Uniqueness #3

For p, q, f, continuous on an open interval I, then

$$y'' + p(x)y' + q(x)y = f(x)$$

has a unique family of solutions on I.

Then, given a value of $x = a$ for the differential equation, there may or may not be a unique solution on the interval.

3.1.2 Linear Independence

A set of functions are mutually linearly independent if it cannot be formed through linear combinations of other functions within the set. In other words, for a set of functions $\{y_1, y_2, y_3, \cdots, y_n\}$,

$$C_1y_1 + C_2y_2 + \cdots + C_ny_n = 0$$

for all x in I and all real C_i.

An nth order linear differential equation is guaranteed to have n linearly independent solutions.

To confirm linear independence, we can use the Wronskian determinant:

$$W(f_1, f_2, \cdots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n)} & f_2^{(n)} & \cdots & f_n^{(n)} \end{vmatrix}$$

If W evaluates to 0 on some interval I, then the functions are linearly dependent. If W evaluates to a function that is 0 nowhere on I, then the functions are linearly dependent.

4 Constant Coefficients

4.1 Homogeneous Equations

A homogeneous differential equation is one of the form

$$L(y) = 0$$

For a homogeneous differential equation, given that y_1 is a solution, any cy_1 is also a solution. In addition, any linear combination of solutions to $L(y) = 0$ will also be solutions. For a differential equation where the coefficients in front of each $y^{(i)}$ are constants ($ay'' + by' + c = 0$), we may "guess"

$$y = e^{rx}$$

By plugging in y and its derivatives into the equation, we end up with a standard quadratic equation (known in general as a characteristic equation):

$$ar^2 + br + c = 0$$

There are three possible cases with the roots (r_1, r_2) resulting from this equation.

If the roots are real and distinct, then $e^{r_1 x}$ and $e^{r_2 x}$ are the two linearly independent solutions to the equation.

If the roots are complex and distinct ($r = p \pm qi$), then

$$\begin{aligned} y &= e^{(p+qi)x} \\ &= e^p e^{iqx} \\ &= e^p(\cos(qx) + i\sin(qx)) \end{aligned}$$

The two linearly independent solutions are thus $e^p \cos(qx)$ and $e^p \sin qx$.

If $r_1 = r_2 = r$, then our two solutions will be e^{rx} and xe^{rx}. In generality, if our characteristic equation has roots of some multiplicity m, then we will multiply e^{rx} with each function from the set $\{x^0, x^1, \cdots, x^{m-1}\}$.

4.2 Non-homogeneous Equations

Given a non-homogeneous equation

$$L(y) = f(x)$$

every solution takes the form

$$y_p + y_c$$

where y_p is a particular solution to the given non-homogeneous, and y_c is the general solution to the associated homogeneous. y_c can just be found by following the method outlined in 3.2.1.

To find y_p, we can use the method of undetermined coefficients.

4.2.1 Method of Undetermined Coefficients

For each term in $f(x)$, we can account for its entire family of derivatives by placing unknown coefficients in front of them. For example, for

$$f(x) = x^2$$

we can guess

$$y_p = Ax^2 + Bx + C$$

By plugging this into the original differential equations, we can solve for the coefficients to find y_p.

If any term of $f(x)$ coincides with a term inside of y_c, we must instead insert that term multiplied by x until there is no overlap between y_c and y_p.

Note that this method cannot be used for an $f(x)$ with a continuing family of derivatives, like $\tan x$.

4.2.2 Variation of Parameters

For non-homogeneous differential equations for which we cannot use the method of undetermined coefficients, we may use the variation of parameters instead. This method, while often more tedious than using undetermined coefficients, can always be used to find a particular solution to a linear nonhomogeneous differential equation, given that we know the general complementary solution y_c to its associated homogeneous. For some non-homogeneous, *monic* differential equation

$$L(y) = f$$

we assume that the particular solution will be of the form

$$y_p = u_1(x)y_1 + u_2(x)y_2 + \cdots + u_n(x)y_n$$

as each component $u_i(x)y_i$ would be independent from the component in the complementary solution C_iy_i. For brevity, we will consider the $n = 3$ case. By applying the product rule n times to our assumed y_p get a system of n equations, we arrive at the system

$$\begin{aligned} u_1'y_1 + u_2'y_2 + u_3'y_3 &= 0 \\ u_1'y_1' + u_2'y_2' + u_3'y_3' &= 0 \\ u_1'y_1'' + u_2'y_2'' + u_3'y_3'' &= 0 \end{aligned}$$

This can be written in matrix form as

$$\begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix}$$

Note that, in the general case, this will be of the form

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ f \end{bmatrix}$$

Consider that the matrix of coefficients for the system based on $u_1,\ u_2,\ u_3$ is the matrix containing $y_1,\ y_2,\ y_3$ and its derivatives, for which the determinant is the Wronskian $W(y_1, y_2, y_3)$. We know that $y_1,\ y_2,\ y_3$ are linearly independent, and therefore, the Wronskian is never zero. Therefore, the matrix containing each y_i and its derivatives has an inverse. Using Cramer's rule to solve the system, we arrive that the conclusion

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ f & y_2'' & y_3'' \end{vmatrix}}{W}$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & f & y_3'' \end{vmatrix}}{W}$$

$$u_3' = \frac{\begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & f \end{vmatrix}}{W}$$

Using these results, we can substitute them back into our initial guess

$$y_p = u_1y_1 + u_2y_2 + u_3y_3$$

5 Non-constant Coefficients

5.1 Existence & Uniqueness #3

For functions p, q, and f, continuous on some interval I,

$$y'' + p(x)y' + q(x)y = f(x)$$

has unique solutions on the interval. Notice that the equation in question must be *monic*.

5.2 Existence & Uniqueness #4

This is an extension to E&U #3, but for nth order differential equations.

5.3 Euler-Cauchy form

For equations of some form

$$x^ny^{(n)} + x^{n-1}y^{(n-1)} + \cdots + xy' + y = 0$$

Our usual guess of e^{rx} will not work. Instead, we guess

$$y = x^r$$

as its successive derivatives involve lower powers of x, which will cancel out with the coefficients of each term. We end up with some polynomial involving just rs, for which we can solve and find our values. If there are duplicate roots to the resulting polynomial, we multiply one of the x^r solutions with $\ln x$.

For complex roots, the process is slightly more difficult. If our roots are of the form

$$r_{1,2} = a \pm bi$$

We end up with a solution x^{a+bi}, for example. To make this non-complex, we do some manipulations.

$$\begin{aligned} x^{a+bi} &= e^{(a+bi)\ln x} \\ &= e^{a\ln x} e^{bi\ln x} \\ &= e^{a\ln x}(\cos(b\ln x) + i\sin(b\ln x)) \\ &= x^a\cos(b\ln x) + ix^a\sin(b\ln x) \end{aligned}$$

Thus, our solutions are

$$x^a\cos(b\ln x)$$

and

$$x^a\sin(b\ln x)$$

6 Reduction of Order

Reducing the order of a differential equation is most helpful for second order equations, for which a single substitution will make it first-order and thus applicable to the convenient methods afforded to first-order equations.

6.1 Substitution #1

For a differential equation of the form

$$F(y^{(n)}, y^{(n-1)}, \ldots, y'', y', x) = 0$$

in which there is no singular y term, we can substitute in the function

$$p(x) = y'$$

Every successive derivative of y with respect to x is also a derivative of p. The differential equation is then converted into something of the form

$$F(p^{(n-1)}, p^{(n-2)}, \dots, p', p, x) = 0$$

6.2 Substitution #2

For a differential equation of the form

$$F(y^{(n)}, y^{(n-1)}, \dots, y'', y', y) = 0$$

in which there is no singular x term, we can substitute in the function

$$p(y) = y'$$

Notice that p is now in terms of the variable y. To find y'',

$$y'' = \frac{d}{dx}(y') = \frac{d}{dx}p(y)$$

By the chain rule,

$$\frac{d}{dx}p(y) = \frac{dp}{dy} \cdot \frac{dy}{dx} = p'y' = p'p$$

This sub often makes equations separable.

7 Linear Systems of Differential Equations

7.1 Definitions

A coupled system of differential equations come in some form

$$f(t, x, y, x', y') = 0$$

$$g(t, x, y, x', y') = 0$$

where t stands as the independent variable to the dependent variables x, y, and their derivatives. Oftentimes, we will express linear differential equations as systems of first order, coupled differential equations instead. This will be of some form

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

Or, for brevity,

$$\mathbf{x}' = \mathbb{A}\mathbf{x} + \mathbf{f}$$

Each solution to this equation, $\mathbf{x}_n$, is linearly independent from the others.

The fundamental matrix of solutions for a homogeneous equation is defined as the matrix holding the entries of each $\mathbf{x}_n$ in its columns.

$$\mathbf{\Phi} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \\ | & | & \cdots & | \end{bmatrix}$$

The Wronskian of the system is just the determinant of $\mathbf{\Phi}$.

7.2 Homogeneous Systems

For a linear, homogeneous system

$$\mathbf{x}' = \mathbb{A}\mathbf{x}$$

with n equations, there will exist n solutions. Before we begin, we note that the principles of superposition still hold in these systems. Similar to how we dealt with scalar linear functions, we may provide a "guess" for the solution to this equation:

$$\mathbf{x} = \mathbf{v}e^{\lambda t}$$

Then,

$$\mathbf{x}' = \lambda\mathbf{v}e^{\lambda t}$$

Substituting back into the original equation,

$$\lambda\mathbf{v}e^{\lambda t} = \mathbb{A}\mathbf{v}e^{\lambda t}$$

$$\lambda\mathbf{v} = \mathbb{A}\mathbf{v}$$

This λ and this $\mathbf{v}$ are known as an eigenvector-eigenvalue pair. This means that, for some linear transform $\mathbb{A}$, the vector $\mathbf{v}$ will only be scaled by some amount; its direction will not change. Through some rearrangement, we arrive at the equation

$$(\mathbb{A} - \lambda I)\mathbf{v} = \mathbf{0}$$

In order to find these eigenvector-eigenvalue pairs, we first solve for the eigenvalue by setting the determinant of the resulting linear transform to zero:

$$\det(\mathbb{A} - \lambda I) = 0$$

For each resulting solution λ, we apply the modified transform to a vector with arbitrary values to solve for relations between the values. It should be noted that eigenvectors are never unique — there is usually one degree of freedom, representing the scaling of the vector. However, there are some cases to consider, like we needed to with scalar linear differential equations.

7.2.1 Repeated Eigenvalues

With repeated eigenvalues, our first solution is simply

$$\mathbf{x}_1 = \mathbf{v}_1 e^{\lambda t}$$

However, with each successive repeated eigenvalue up to some multiplicity k,

$$\mathbf{x}_2 = \mathbf{v}_1 t e^{\lambda t} + \mathbf{v}_2 e^{\lambda t}$$

where $\mathbf{v}_2$ is the solution to

$$(\mathbb{A} - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$$

Subsequently,

$$\mathbf{x}_2 = \mathbf{v}_1 t^2 e^{\lambda t} + \mathbf{v}_2 t e^{\lambda t} + \mathbf{v}_3 e^{\lambda t}$$

where

$$(\mathbb{A} - \lambda I)\mathbf{v}_3 = \mathbf{v}_2$$

7.2.2 Complex Eigenvalues

If we end up with some pair of complex eigenvalues

$$\lambda = a \pm bi$$

we can essentially solve through for the eigenvectors, finding complex values for each of the entries. We may use Euler's formula to expand the $e^{(a+bi)t}$ term, and express our eigenvector as linear combination of real and complex components. Then, we can expand the resulting expression to find the real and complex components of the entire expression, both of which serve as solutions to the equation.

7.3 Non-homogeneous Systems

7.3.1 Method of Undetermined Coefficients

For an $\mathbf{f}$ of some form

$$\mathbf{a}f(x)$$

where $f(x)$ has a terminating tree of derivatives, we can use the method of undetermined coefficients. Once we have found our complementary solution, $\mathbf{x}_c$, we can "guess" our particular solution, $\mathbf{x}_p$, by setting up some unknown coefficient vectors and plugging them into the equation, just as we did in a scalar system. For example, an $\mathbf{f}$ of the form

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} x$$

would yield a guess of

$$\mathbf{x}_p = \mathbf{a}x + \mathbf{b}$$

which would then be plugged in to the original equation to solve for $\mathbf{a}$ and $\mathbf{b}$. If any part of our $\mathbf{x}_p$ coincides with a term in $\mathbf{x}_c$, we need to revise our guess, multiplying that specific term by t and accounting for additional derivatives.

7.3.2 Variation of Parameters

When we obtain the $\mathbf{\Phi}$ to the associated homogeneous equation of our differential equation, we will guess that the particular solution is of the form

$$\mathbf{x}_p = \mathbf{\Phi}\mathbf{v}(t)$$

When this is plugged into the system, we arrive at the conclusion

$$\mathbf{\Phi}'\mathbf{v} + \mathbf{\Phi}\mathbf{v}' = \mathbb{A}\mathbf{\Phi}\mathbf{v} + \mathbf{f}$$

This can be rewritten as

$$\mathbf{\Phi}\mathbf{v}' = \mathbf{f}$$

Because each entry in $\mathbf{\Phi}$ is linearly independent, its determinant cannot be 0, and it is thus an invertible matrix. Therefore,

$$\mathbf{v}' = \mathbf{\Phi}^{-1}\mathbf{f}$$

$$\mathbf{v} = \int \mathbf{\Phi}^{-1}\mathbf{f}dt$$

Therefore,

$$\mathbf{x}_p = \mathbf{\Phi} \int \mathbf{\Phi}^{-1}\mathbf{f}dt$$

It can thus be seen that the general form of a solution to a non-homogeneous system is

$$\mathbf{x} = \mathbf{\Phi}\mathbf{c} + \mathbf{\Phi} \int \mathbf{\Phi}^{-1}\mathbf{f}dt$$

where $\mathbf{c}$ is a vector of arbitrary constants.

7.4 Initial Value Problems

An IVP in such a system has the constraints

$$\mathbf{x}(a) = \mathbf{b}$$

where the unknown is $\mathbf{c}$, the vector of arbitrary constants that generates a linear combinations of solutions contained in $\mathbf{\Phi}$. Recall

$$\mathbf{x} = \mathbf{\Phi}\mathbf{c} + \mathbf{\Phi}\mathbf{u}$$

We find this $\mathbf{c}$ to be equal to

$$\mathbf{c} = (\mathbf{\Phi}(a))^{-1}\mathbf{b}$$

while

$$\mathbf{u} = \int_a^t \mathbf{\Phi}^{-1}(s)\mathbf{f}(s)ds$$

where $\mathbf{u}$ has been converted from an indefinite integral to an accumulator function beginning at the point a.

7.5 Matrix Equations

Recall the fundamental matrix of solutions $\mathbf{\Phi}$ for a linear differential equation, like

$$\mathbf{x}' = \mathbb{A}\mathbf{x}$$

We may construct from this a square matrix

$$\mathbf{X} = \mathbf{\Phi}(t)$$

that is the solution to the *matrix differential equation*

$$\mathbf{X}' = \mathbb{A}\mathbf{X}$$

7.5.1 Matrix Exponentials

For a scalar differential equation like

$$y' = ay$$

the obvious solution is that $y = e^{ax}$. If we apply this relation to our matrix equation

$$\mathbf{X}' = \mathbb{A}\mathbf{X}$$

then we would arrive at a solution like

$$\mathbf{X} = e^{\mathbb{A}t}$$

But what does it even mean to exponentiate a function to a matrix?

Recall the Taylor series expansion of the function e^x.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

If we applied this to our matrix exponentiation, we would arrive at

$$e^{\mathbb{A}} = \mathbf{I} + \mathbb{A} + \frac{\mathbb{A}^2}{2!} + \frac{\mathbb{A}^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{\mathbb{A}^n}{n!}$$

Suppose $\mathbb{A}$ is a diagonal matrix:

$$\mathbb{A} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

Note that for an $n \times n$ diagonal matrix,

$$\begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix}^k = \begin{bmatrix} a_1^k & 0 & \cdots & 0 \\ 0 & a_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n^k \end{bmatrix}$$

then

$$\begin{aligned} e^{\mathbb{A}} &= \mathbf{I} + \mathbb{A} + \frac{\mathbb{A}^2}{2!} + \cdots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} \frac{a^2}{2!} & 0 \\ 0 & \frac{b^2}{2!} \end{bmatrix} + \cdots \\ &= \begin{bmatrix} 1 + a + \frac{a^2}{2!} + \cdots & 0 \\ 0 & 1 + b + \frac{b^2}{2!} \end{bmatrix} \\ &= \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix} \end{aligned}$$

Therefore,

$$e^{\mathbb{A}t} = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{bmatrix}$$

7.5.2 Diagonalization

What if $\mathbb{A}$ is not a diagonal matrix?

Every square matrix can be expressed as

$$\mathbb{A} = \mathbf{PDP}^{-1}$$

where

$$\mathbf{P} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & \cdots & | \end{bmatrix}$$

and

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

where each corresponding eigenvalue-eigenvector pair corresponds to the same column in each matrix. Now, we can try exponentiating this re-expression.

$$\begin{aligned}
\mathbb{A} &= \mathbf{PDP}^{-1} \\
\mathbb{A}^k &= \left(\mathbf{PDP}^{-1}\right)^k \\
&= (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1})\cdots(\mathbf{PDP}^{-1}) \\
&= \mathbf{PD}(\mathbf{P}^{-1}\mathbf{P})\mathbf{D}(\mathbf{P}^{-1}\mathbf{P})\mathbf{D}\cdots(\mathbf{P}^{-1}\mathbf{P})\mathbf{DP}^{-1} \\
&= \mathbf{PD}^k\mathbf{P}^{-1}
\end{aligned}$$

Note that $\mathbf{D}$ is a diagonal matrix. Therefore,

$$\mathbb{A}^k = \mathbf{P}\begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}\mathbf{P}^{-1}$$

Applying this identity,

$$\begin{aligned}
e^{\mathbb{A}t} &= e^{\mathbf{PDP}^{-1}t} \\
&= \mathbf{I} + \mathbf{PDP}^{-1}t + \frac{\mathbf{PD}^2\mathbf{P}^{-1}t^2}{2!} + \cdots \\
&= \mathbf{PP}^{-1} + \mathbf{PDP}^{-1}t + \frac{\mathbf{PD}^2\mathbf{P}^{-1}t^2}{2!} + \cdots \\
&= \mathbf{P}\left(\mathbf{I} + \mathbf{D}t + \frac{\mathbf{D}^2t^2}{2!} + \cdots\right)\mathbf{P}^{-1} \\
&= \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1}
\end{aligned}$$

8 Laplace Transforms

8.1 Definition

A Laplace transform takes a function of the independent variable t and transforms it into a function of s.

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt = F(s)$$

The inverse Laplace transform goes in the other direction:

$$\mathcal{L}\{F(s)\} = f(t)$$

Every function f has a unique transfrom F. We are thus guaranteed the existence of a unique inverse.

The Laplace transform is a linear operation.

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

Laplace transforms also work on piecewise continuous functions. We just need to break up the integral in accordance with the pieces.

The power of Laplace transforms lie in their ability to turn derivatives, such as in the case of differential equations, into algebraic problems.

8.2 Initial Value Problems

If we apply a Laplace transform to some nth derivative of a function, we receive an output involving the values of the function and its derivatives at 0 and the Laplace transform of the original function f.

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - \sum_{i=0}^{n-1} s^{n-1-i} f^{(i)}(0)$$

If we apply the transform to some linear initial value problem of an nth order differential equation, we will receive a function involving the Laplace transform of the solution to the equation (F), s, and constants. By isolating and solving for F, we can then perform an inverse transform to find the answer.

Given some resulting F with a denominator that is factorable into simpler expressions, we can use partial fractions to separate the fraction.

$$X(s) = \frac{3}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

The resulting forms are similar to the transforms for sin and cos.

8.3 Translation

A translation of a function in the s domain results in an e^{at} factor when the inverse transform is applied.

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$
$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$$

On the other hand, translating a function on the t axis involves the Heaviside function. For a function shifted to the right by a units, the resulting translation is of the form

$$u(t-a)f(t-a)$$

The Laplace transform of this function is

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$$

8.4 Convolution

The convolution $f * g$ of two functions is defined to be

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

Notably, with a convolution,

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\} \cdot \mathcal{L}\{g\}$$

This operation is commutative:

$$f * g = g * f$$

Given two transforms F and G, then,

$$\mathcal{L}^{-1}\{F(s) \cdot G(s)\} = f(t) * g(t)$$
$$f(t) * g(t) = \int_0^t \mathcal{L}^{-1}\{F(s)\}\Big|_{t-\tau} \mathcal{L}^{-1}\{G(s)\}\Big|_{\tau} d\tau$$

8.5 Periodic Functions

For some periodic function $f(t)$ with a period p, its transform will be of the form

$$F(s) = \frac{1}{1-e^{-ps}} \int_0^p e^{-st}f(t)dt$$

9 Series Solutions

Recall that all functions can be expressed as a power series of polynomials, via Taylor series. Given a differential equation, we can make a “guess” as to the solution:

$$y = \sum_{n=0}^{\infty} c_n x^n$$

where each c_n is an arbitrary constant (which we hope to match to the Taylor series of an existing function later). We can take derivatives of the series:

$$\begin{aligned} y &= \sum_{n=0}^{\infty} c_n x^n \\ y' &= \sum_{n=1}^{\infty} n c_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} \\ &\vdots \\ y^{(k)} &= \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} x^{n-k} \end{aligned}$$

and substitute them for each $y^{(n)}$ in our original equation.

This will yield a complicated sum of sums starting at different indices, with each x to a different power.

To remedy this, we can undergo two transformations: termed ‘tack’ and ‘shift’.

9.1 Tack

‘Tacking’ refers to adding and then subtracting the missing terms of the series so that we can start with a lower index. For example, with a series like

$$\sum_{n=1}^{\infty} n c_n x^{n-1}$$

we can add and then subtract the $n = 0$ case:

$$\sum_{n=1}^{\infty} nc_n x^{n-1} + 0 \cdot c_0 x^{-1} - 0 \cdot c_0 x^{-1} = \sum_{n=0}^{\infty} nc_n x^{n-1} - 0$$

Notice that the power of x remains the same, while the starting index is moved over.

9.2 Shift

'Shifting' refers to changing n to equal something like $n + 1$ so that we shift the entire series over by one index. For the same example

$$\sum_{n=1}^{\infty} nc_n x^{n-1}$$

a shift would transform the series into

$$\sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$$

This changes both the starting index and the power of x.

9.3 Solving

Given some equation (first order, for brevity)

$$y' + y = f(x)$$

we combine the series of $y' + y$ after transforming their starting indices and powers of x to be the same. We then match each coefficient of the resulting sum of the series to each coefficient in the Taylor series expansion for $f(x)$.

For example, for some equation

$$y' + y = 0$$

The series of $y' + y$ would be equal to

$$\sum_{n=0}^{\infty} ((n+1) c_{n+1} + c_n) x^n = 0$$

following a shift in the series for y'. By the Taylor series expansion for the constant function 0, then,

$$(n+1)c_{n+1} + c_n = 0$$

for all n. We then see that

$$c_{n+1} = -\frac{c_n}{n+1}$$

We can plug multiple values in for n to guess a pattern for each c_n, which will determine the final series.