Physics C: Mechanics

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April 15, 2023

Contents

1 Kinematics

1.1 1-D

1.1.1 Definitions

note: all expressions here rely on acceleration being a constant value.

The *instantaneous velocity* of a particle at some point is defined to be the derivative of the path evaluated at the specified point.

$$v(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = x'(t) = \frac{dx}{dt}$$

In addition, the distance traversed by some particle is given by the definite integral of its velocity function:

$$\Delta x = \int_{t_1}^{t_2} v(t)dt$$

The *instantaneous speed* is the magnitude of the velocity vector:

$$|v(t)|$$

The *instantaneous acceleration* is defined as the change in velocity over time:

$$a(t) = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = x''(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

Analogously,

$$\Delta v = \int_{t_1}^{t_2} a(t)dt$$

The *average velocity*, meanwhile, is the total distance traversed divided by the total time taken.

$$\bar{v} = \frac{\Delta x}{\Delta t}$$

Similarly, the *average acceleration*, is defined as

$$\bar{a} = \frac{\Delta v}{\Delta t}$$

1.1.2 "Big-Five" Equations

These equations, which can be derived from algebraic manipulations, can be used to solve for unknown variables in any system.

$$\begin{aligned}
\Delta x &= \bar{v}t \\
v_f &= v_0 + at \\
x &= v_0 t + \frac{1}{2}at^2 \\
x &= v_f t - \frac{1}{2}at^2 \\
v_f^2 &= v_0^2 + 2a(\Delta x)
\end{aligned}$$

Notice how each of these equations has a specific variable that does not show up, which allows for the solving of that variable through other means and substitutions.

1.1.3 Free Fall

Free fall problems are simply just a special case of 1-D motion, except that a is always equal to the acceleration due to gravity on the Earth's surface, 9.8. However, this is frequently rounded to 10 $\frac{\mathrm{m}}{\mathrm{s}^2}$. In many problems, the downwards direction is assigned a *negative* value.

1.2 2-D

1.2.1 Definitions

note: all expressions here rely on acceleration being a constant value.

Position, velocity and acceleration are vector quantities: to describe one such quantity, you need both a magnitude and a direction. Mathematicallly, we define a position vector as a linear combination of the basis vectors **i**, **j**, **k** and so forth (although **k** is only for 3-D situations):

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}$$

Therefore, we can integrate and differentiate with respect to these vector quantities to find velocity and acceleration.

$$\mathbf{v}(t) = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt}$$

$$\mathbf{a}(t) = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$$

And analogously to 1-D motion, we can integrate quantities to obtain distance traversed and the change in velocity.

$$\Delta \mathbf{r} = \int_{t_1}^{t_2} \mathbf{v}(t) dt$$

$$\Delta \mathbf{v} = \int_{t_1}^{t_2} \mathbf{a}(t) dt$$

Derivatives and integrals are *component-wise* operations, meaning that they are applied to each component of the vectors separately. What is really meant by the expression $\frac{d\mathbf{r}}{dt}$, for example, is

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$$

Because these are vector quantities, finding the speed or magnitude of displacement requires the use of the Euclidean distance formula.

$$|\mathbf{r}| = \sqrt{x^2 + y^2}$$

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2}$$

Now, because taking derivatives and integrals are component-wise, this means that quantities in the x and y direction are essentially independent; we do not have to worry about one influencing the other. This means that when solving for unknown quantities, we can simply "solve in one direction". Thus, a system of 2-D kinematics is really just two separate 1-D systems in a trenchcoat.

1.2.2 "Big five" extended

The equations from 1-D kinematics still hold true in 2-D systems.

$$\begin{aligned}
\Delta\mathbf{r} &= \bar{\mathbf{v}}t \\
\mathbf{v}_f &= \mathbf{v}_0 + \mathbf{a}t \\
\mathbf{r} &= \mathbf{v}_0 + \frac{1}{2}\mathbf{a}t^2 \\
\mathbf{r} &= \mathbf{v}_f - \frac{1}{2}\mathbf{a}t^2 \\
\mathbf{v}_f^2 &= \mathbf{v}_0^2 + 2\mathbf{a}\mathbf{r}
\end{aligned}$$

1.2.3 Solving systems

Many problems will ask about whether or not two particles will collide, based on their position vectors. In these situations, a particle will only collide with another if there is a single value of t that satisfies both systems of equations:

$$x_1(t) = x_2(t)$$

$$y_1(t) = y_2(t)$$

where $x_1,\ x_2,\ y_1,\ y_2$ are functions of position based on time. To find angles between quantities, we can leverage the dot product, defined to be

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i \cdot b_i = |\mathbf{a}||\mathbf{b}| \cos\theta$$

1.2.4 Projectile Motion

Recall the conversions between the polar form of a path (given angle of elevation and a magnitude) and the rectangular form (given x and y components of a vector).

$$v = \sqrt{v_x^2 + v_y^2}$$

$$\theta = \arctan\frac{v_y}{v_x}$$

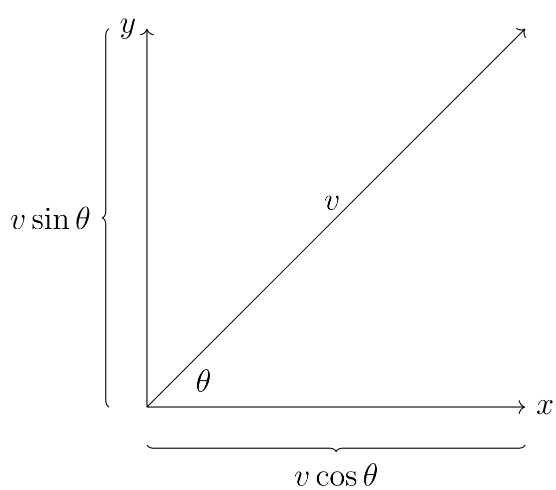
For simplicity, we will assume that the only force acting on the system is the force of gravity.

In the x direction, velocity is constant because there is no horizontal force acting on the system.

$$\begin{aligned} v_x(t) &= v_{x0} \\ x(t) &= x_0 + v_x t \\ v_{x0} &= v_0 \cos\theta \end{aligned}$$

In the y direction, meanwhile, we need to consider g, the acceleration due to gravity on Earth.

$$\begin{aligned} a_y &= -g \\ v_y &= v_{y0} - gt \\ \Delta y &= v_{y0}t - \frac{1}{2}gt^2 \\ \Delta y &= v_{f0}t + \frac{1}{2}gt^2 \\ v_y^2 &= v_{y0}^2 - 2g(\Delta y) \\ v_{y0} &= v_0 \sin\theta \end{aligned}$$



At the peak of some trajectory, v_y will be equal to 0 as this is the point where the downwards velocity due to gravity overtakes the initial upwards velocity. All simple projectile motion will be parabolic, which means that we can leverage symmetry to solve some situations.

Projectile Equations

Through algebraic manipulations of our initial kinematics laws for projectile systems, we can arrive at equations for the time and range of a projectile.

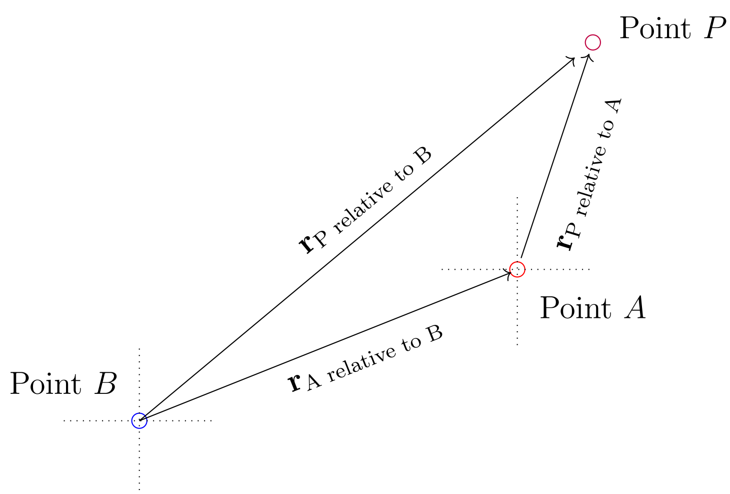
$$t_{\text{total}} = \frac{2v_0 \sin\theta}{g}$$

$$R \text{ (range)} = \frac{v_0^2 \sin 2\theta}{g}$$

$$y = \tan\theta \cdot x - \left(\frac{g}{2(v_0 \cos\theta)^2}\right) \cdot x^2$$

1.2.5 Relative Quantities

When two objects are in relative motion, we need to decide a frame of reference with which to analyze the system. Positions will be meaningless unless we specify where the origin of the coordinate plane is, for example. We can examine relative position:



From here, we can observe a relationship of the position vector of point P relative to point B:

$$\mathbf{r}_{\text{P relative to B}} = \mathbf{r}_{\text{P relative to A}} + \mathbf{r}_{\text{A relative to B}}$$

We can further extend this relationship by taking derivatives to yield relationships between velocities and accelerations.

$$\mathbf{v}_{\text{P relative to B}} = \mathbf{v}_{\text{P relative to A}} + \mathbf{v}_{\text{A relative to B}}$$

$$\mathbf{a}_{\text{P relative to B}} = \mathbf{a}_{\text{P relative to A}} + \mathbf{a}_{\text{A relative to B}}$$

Inertial reference frames move with a constant velocity with respect to each other, meaning that

$$\mathbf{a}_{\text{B relative to A}} = 0$$

which then means

$$\mathbf{a}_{\text{P relative to B}} = \mathbf{a}_{\text{P relative to A}}$$

1.2.6 Uniform Circular Motion (UCM)

In circular motion, we have two components of acceleration: the tangential component, which controls the magnitude of the velocity, and the radial component, which controls the direction of the velocity. Note that $a_{tan} = \frac{dv}{dt}$. From simple trigonometry, we can gather that the position equation of a point undergoing UCM is

$$\mathbf{r} = \langle r\cos\theta, r\sin\theta \rangle$$

To find θ in terms of some time parameter t, we introduce a new parameter ω which represents angular speed ($\frac{v}{r}$). Then,

$$\theta = \omega t$$

In order to account for initial angular position, we introduce another parameter ϕ that indicates the initial angle of the particle. Thus,

$$\theta = \omega t + \phi$$

and the position equation becomes

$$\mathbf{r}(t) = \langle r\cos(\omega t + \phi), r\sin(\omega t + \phi) \rangle$$

We can derive vector equations for both the velocity and acceleration from this position equation by taking multiple derivatives.

$$\mathbf{v}(t) = \mathbf{r}'(t) = \omega r \langle -\sin(\omega t + \phi), \cos(\omega t + \phi) \rangle$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = -\omega^2 r \langle \cos(\omega t + \phi), \sin(\omega t + \phi) \rangle$$

Notice how the acceleration and position vectors are scalar multiples of each other, while the velocity vector is orthogonal to both. The magnitude of the acceleration vector, by using the pythagorean theorem, comes out to

$$|\mathbf{a}(t)| = \omega^2 r = \frac{v^2}{r}$$

2 Dynamics

2.1 Newton's Laws

1.
An object at rest will remain at rest until an external force acts on it. An object in motion will remain at a constant speed and direction unless some unbalanced force affects it.
2.

$$\mathbf{F} = m\mathbf{a}$$

3.
Any force exerted by one object has an equal and opposite force exerted on the object.

2.2 Forces

The superposition of all forces on a system is known as the net force of the system.

$$\mathbf{F}_{\text{net}} = \sum_{i=1}^{n} \mathbf{F}_i$$

This is a useful principle because it allows us to look at the total effects of a system and break forces down into components in order to solve problems. In practical solving, we usually break forces down into their x and y components.

$$\mathbf{F} = \langle F_x, F_y \rangle$$

$$F_{\text{net (x)}} = \sum_{i=1}^{n} F_{i\text{ (x)}}$$

$$F_{\text{net (y)}} = \sum_{i=1}^{n} F_{i\text{ (y)}}$$

Inertia is the tendency of an object to resist acceleration. Mass is a measure of inertia, while weight is a measure of force.

2.2.1 Types of Forces

Gravitational Force

Gravitational force is going to be the acceleration due to gravity with respect to a surface times the mass.

$$F_g = mg$$

Normal Force

The normal force of an object against a surface is always perpendicular to the surface. For example, an object on a horizontal surface with no external forces acting on it will have a normal force with magnitude mg. If some additional vertical downwards force F_0 were to be applied to it, the normal force of the object would be $mg + F_0$. On an inclined plane, this is equal to $mg\cos\theta$ where θ is the angle of the plane to the horizontal.

Frictional Force

To calculate the frictional force, we need something known as the coefficient of friction, symbolized μ. Friction forces are divided into two categories: static and kinetic. Static friction refers to the amount of force required to make an object move at first, while kinetic friction refers to the friction encountered by an object while moving on a surface. In general, the frictional force is the product of the coefficient of friction and the normal force against the surface.

$$F_k = \mu_k F_N$$

$$F_s = \mu_s F_N$$

Tension Force

The tension force will always be parallel to the rope or string experiencing it. The tension will always pull on the objects it is connected to, so the force will always point towards the middle of the string.

Spring Force

The spring force is governed by Hooke's Law, which states that

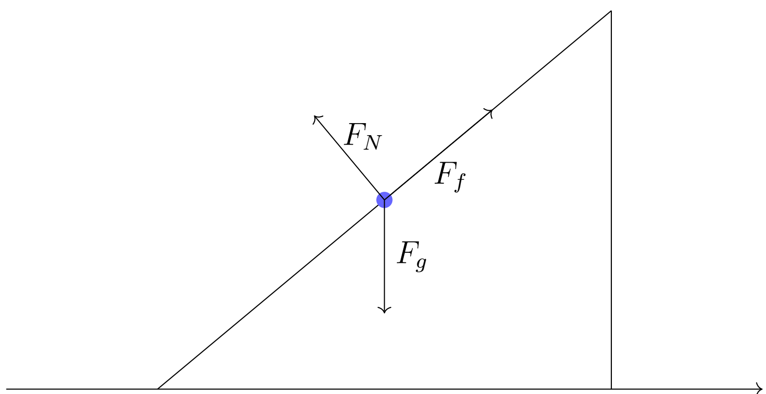
$$F_r = -k\Delta x$$

where F_r denotes the restoring force of the spring, k denotes the specific spring constant, and Δx is the displacement of mass at the end of the spring from its equilibrium position.

2.2.2 Types of Problems

Inclines

The free body diagram of a typical incline problem looks something like



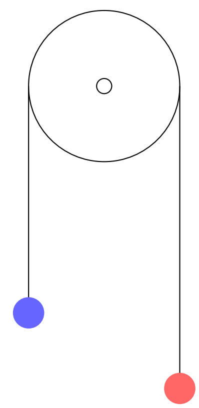
Note that F_g will be equal to mg. The component parallel to the plane, then, is $mg\sin\theta$, and its normal component is $mg\cos\theta$. Then, the magnitude of the normal force and frictional force just fall out.

$$F_N = mg\cos\theta$$

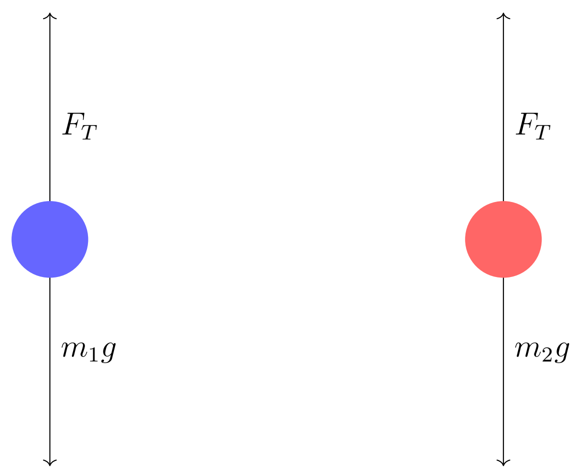
$$F_f = \mu mg\sin\theta$$

Atwood Machines

Atwood machines consist of pulleys and strings, systems of which we can solve in terms of like forces.



In this system, we have two masses, blue and red (we'll call them m_1 and m_2 respectively). We can make free body diagrams for each:



The tension force will always be the same across the entire rope. We can then make a system of equations:

$$F_{\text{net }(m_1)} = F_T - m_1g = m_1a_1$$

$$F_{\text{net }(m_2)} = F_T - m_2g = m_2a_2$$

So we have two equations with three unknown variables ($F_T,\ a_1,\ a_2$). The third equation needed to solve the system completely comes from the idea that the velocities of each mass must be equal in magnitude and opposite in direction because they are connected by a rope.

$$\mathbf{v}_1 = -\mathbf{v}_2$$

By taking a derivative,

$$\mathbf{a}_1 = -\mathbf{a}_2$$

These are the three equations needed to solve for the three unknowns in this system.

3 Energy

3.1 Work

The work done on a single object due to an external force $\mathbf{F}$ is defined to be

$$dW = \mathbf{F} \cdot d\mathbf{r}$$

Note that, when multiple forces act on the object, this is equivalent to saying

$$dW = \mathbf{F}_{\text{net}} \cdot d\mathbf{r} = \left(\sum F_i\right) \cdot d\mathbf{r}$$

Work has units of a joule, which is also known as a newton-meter.

$$J = N \cdot m = \text{kg} \cdot \text{m}^2/\text{s}^2$$

Because work is defined to be a dot product, it is a scalar value. Corresponding to the sign of work, there are three cases.

1. Work is negative — The force has a component antiparallel to $d\mathbf{r}$, the displacement vector. There is some tangential acceleration that causes the object to slow down.

2. Work is zero — The force has no component parallel to $d\mathbf{r}$ and has no tangential acceleration. The speed of the object doesn't change.

3. Work is positive — This can be thought of as the opposite of case 1: the force has a parallel component to $d\mathbf{r}$. There is some tangential acceleration that causes the object to speed up.

While we can reasonable calculate work through a dot product for constant forces, we cannot do so for forces that are a function of the distance traversed. We thus need to implement integration to deal with such cases.

$$W = \int_i^f \mathbf{F} \cdot d\mathbf{r} = \int_{x_i}^{x_f} F_x dx + \int_{y_i}^{y_f} F_y dy$$

where i denotes the initial position and f denotes the final position. Note that force is the derivative with respect to displacement of work:

$$F = \frac{dW}{dx}$$

3.1.1 Work and Speed

Recall the definition of work:

$$\int dW = \int \mathbf{F} \cdot d\mathbf{r}$$

Because the force will not necessarily be constant, we need to rewrite the integrand of the right hand sixde in terms of a variable and its own differential.

$$\mathbf{F} \cdot d\mathbf{r} = m\mathbf{a} \cdot d\mathbf{r} = m(a_x dx + a_y dy) = m\left(\frac{dv_x}{dt}dx + \frac{dv_y}{dt}dy\right)$$

(excuse the abuse of notation here) we can rearrange this result:

$$m\left(\frac{dx}{dt}dv_x + \frac{dy}{dt}dv_y\right) = m(v_x dv_x + v_y dv_y)$$

And now we can integrate:

$$\begin{aligned}
\int dW &= \int m(v_x dv_x + v_y dv_y) \\
W &= m\left(\int v_x dv_x + \int v_y dv_y\right) \\
W &= m\left[\frac{v_x^2}{2} + \frac{v_y^2}{2}\right]_{v_0}^{v_f} \\
W &= \frac{m}{2}[(v_{f\ (x)}^2 + v_{f\ (y)}^2) - (v_{0\ (x)}^2 + v_{0\ (y)}^2)] \\
v &= \sqrt{v_x^2 + v_y^2} \\
W &= \frac{mv_F^2}{2} - \frac{mv_0^2}{2}
\end{aligned}$$

3.2 Energy

3.2.1 Kinetic Energy

Kinetic energy is defined to be

$$KE = \frac{mv^2}{2}$$

Recall from the last section that we derived the relation

$$W = \frac{mv_F^2}{2} - \frac{mv_0^2}{2}$$

The right hand side is simply the change in kinetic energy, or ΔKE. This leads us to an important theorem, known as the *Work-Kinetic Energy Theorem*:

$$W = \Delta KE$$

3.2.2 Potential Energy

Gravitational Potential Energy

The equation for gravitational potential energy is

$$U_{\text{gravitational}} = mgh$$

where m denotes mass, g denotes gravity, and h denotes the height above the surface.

Elastic Potential Energy

Elastic potential energy (that is, potential energy for a spring system) is given by the equation

$$U_{\text{spring}} = \frac{kx^2}{2}$$

where k is the very same spring constant, and x is the distance traversed by the mass at its end.

3.2.3 Conservation of Energy

The principle of the conservation of energy states that energy cannot ever be created, nor destroyed; it can only be converted into different forms. For example, kinetic energy can transform to potential energy, and vice versa.
Conservative forces refer to conversions in energy that are easily reversible. The operation of converting kinetic energy to potential energy can, for example, easily be converted the other way.
Non-conservative forces refer to energy conversions that aren't easily reversible, if virtually impossible. We can convert kinetic energy to sound or heat energy, but the same cannot necessarily be said for the other way around.
When conservative forces perform negative work and remove kinetic energy from an object, that excess energy gets converted to potential energy. An alternate definition of a conservative force is one in which a potential energy function, like $U(x, y, z)$, can be defined such that the work is equal to the change in that function between two different points.

$$W = -\Delta U(x, y, z) = U(x_1, y_1, z_1) - U(x_2, y_2, z_2)$$

This law is fundamental to our understanding of energy, as we can derive all of the equations that express potential energy from this single equation. In addition, notice how we can now say

$$\Delta KE = -\Delta U$$

which makes intuitive sense given physical interactions.

3.2.4 Unifying KE and U

Given situations where multiple conservative forces are acting (gravity and spring forces, for example), we need to find a way to unify these into a single expression.

$$W_{\text{net}} = \sum W_i = \sum(-\Delta W_i)$$

Let W and U denote net work and net potential energy, respectively. Then we end up with the equation

$$W_{\text{net}} = -\Delta U_{\text{net}}$$

Note that this is only valid for cases where all forces are conservative. The mechanical energy of a system is defined as the sum of the kinetic and potential energies.

$$E_{\text{mech}} = KE + U$$

Without nonconservative forces, mechanical energy is constant.

$$\Delta E_{\text{mech}} = \Delta(KE + U) = \Delta KE + \Delta U = W - W = 0$$

With non-conservative forces, however,

$$\Delta E_{\text{mech}} = W_{\text{non-conservative}}$$

And so the general form of the energy conservation equation is expressed as

$$KE_1 + U_1 + W_{\text{non-conservative}} = KE_2 + U_2$$

3.3 Power

Power is defined to be the rate at which a force does work on a system.

$$P = \frac{dW}{dt}$$

$$\bar{P} = \frac{W}{\Delta t}$$

Power has units of joules per second, which is the definition of a watt. For constant forces,

$$P = \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \mathbf{F} \cdot \mathbf{v}$$

4 Linear Momentum and Centroids

4.1 Momentum and Impulse

Momentum is a vector quantity, defined as

$$\mathbf{p} = m\mathbf{v}$$

Force is the time derivative of momentum: that is,

$$\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt} = \frac{d\mathbf{p}}{dt}$$

When the net force on a system is 0, the momentum of the system remains constant.

$$\sum_{\text{system } (t_1)} \mathbf{p} = \sum_{\text{system } (t_2)} \mathbf{p}$$

The impulse of a system is defined to be

$$\mathbf{J} = \int_{t_1}^{t_2} \mathbf{F} dt$$

For constant forces, this means that

$$\mathbf{J} = \mathbf{F}\Delta t$$

Impulse is also defined as the change in momentum:

$$d\mathbf{p} = \mathbf{F}dt \Rightarrow \mathbf{J} = \int d\mathbf{p} = \Delta\mathbf{p}$$

Recall the integral mean value theorem:

$$\text{avg}(f(x)) = \frac{1}{b-a} \int_a^b f(x)dx$$

This is applied to forces in terms of momentum, so

$$\mathbf{F}_{\text{average}} = \frac{\int_{t_1}^{t_2} \mathbf{F}dt}{t_2 - t_2} = \frac{\mathbf{J}}{\Delta t}$$

4.2 Collisions

There are three (really, two) types of collisions.

1. In *elastic collisions*, kinetic energy is conserved, meaning that it is the total kinetic energy is the same before and after the collision.

2. In *inelastic collisions*, kinetic energy is not necessarily conserved, and energy may be converted into other forms.

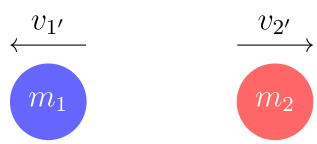
3. In *totally inelastic collisions*, kinetic energy is not conserved and the colliding objects remain stuck together following the collision.

Momentum is generally conserved in all three types of collisions, given that there is no intervention of an external force.

4.2.1 Elastic collisions

The basic makeup of an elastic collision looks something like

After these balls collide,



We have multiple pieces of information available. We know that momentum was conserved, so

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{v}_{1'} + m_2\mathbf{v}_{2'}$$

We also know that kinetic energy in the system is conserved, and thus

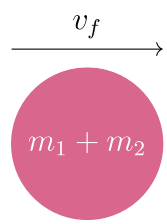
$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v_{1'}^2 + \frac{1}{2}m_2v_{2'}^2$$

This system of equations provides us with a way to find unknown symbols through given quantities.

4.2.2 Inelastic collisions

With inelastic collisions, we have two types: sticky and non-sticky (or totally inelastic vs. not). In a totally inelastic collision, the masses stick together afterwards.

becomes



Notice how we can leverage the idea that the masses move together following a totally inelastic collision to unite the velocities. We are left with the equation

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = (m_1 + m_2)\mathbf{v}_f$$

With non-sticky inelastic collisions, we do not have this ability; we are just left with the expression

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{v}_{1'} + m_2\mathbf{v}_{2'}$$

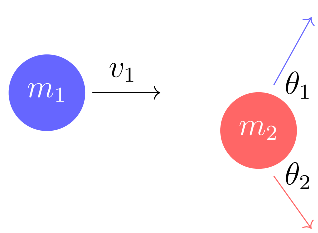
4.2.3 Two-dimensional collisions

Recall that $\mathbf{p}$ is a vector, and thus the conservation of momentum in more dimensions will need to be expressed in terms of a system of two equations.

$$p_x = p'_x$$

$$p_y = p'_y$$

For example, in a collision that looks something like



where the blue ball will bounce off of the red ball, each moving in their respective colored arrows. In such a situation, we must analyze the x and y components of our initial momentum and create a system of equations to follow. Since the blue ball is the only initially moving component of the system,

$$p_x = m_1 v_1$$

$$p_y = 0$$

We know, then, that the end components of the collision must add up to these initial values.

$$p_x = m_1 v_{1'} \cos \theta_1 + m_2 v_{2'} \cos(2\pi - \theta_2) = m_1 v_1$$
$$p_y = m_1 v_{1'} \sin \theta_1 + m_2 v_{2'} \sin(2\pi - \theta_2) = 0$$

4.3 Center of Mass

The center of mass for a given system is defined to be the weighted average of the locations of masses in a system. Intuitively, it is a point on some object or lamina with which one can balance the entire object.

$$\mathbf{r}_{CM} = \frac{\sum m_i \mathbf{r}_i}{\sum m_i} = \frac{\sum m_i \mathbf{r}_i}{M}$$

$$CM_x = \frac{\sum m_i x_i}{M},\ CM_y = \frac{\sum m_i y_i}{M}$$

This naive sum method, however, will only work for discrete point masses. What happens when we have a continuous mass distribution? We can resolve this problem by introducing integrals and differentials.

$$\mathbf{r}_{CM} = \lim_{n\to\infty} \frac{\sum_{i=1}^{i=n} m_i \mathbf{r}_i}{\sum_{i=1}^{i=n} m_i} = \frac{\int \mathbf{r}dm}{\int dm}$$

Now, what is this dm? Intuitively, dm represents a tiny piece of mass that is being added up. Therefore, it makes sense to represent it as some uniform density times the area in question, so we end up with

$$dm = \lambda(x)ds$$
$$dm = \sigma(x)dA$$
$$dm = \rho(x)dV$$

where λ, σ, ρ represent density functions. With multivariable calculus, we can find a more complete set of general formulas for the centroids.

$$M_x = \iint_R y \cdot \rho(x, y)dA$$
$$M_y = \iint_R x \cdot \rho(x, y)dA$$
$$M = \iint_R \rho(x, y)dA$$
$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$$

4.3.1 Moving the CM

Given the equation for the position of the center of mass,

$$\mathbf{r}_{CM} = \frac{\sum m_i \mathbf{r}_i}{M}$$

we can differentiate to find expressions for the velocity, acceleration, momentum, and force of the center of mass.

$$\mathbf{v}_{CM} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}\left(\frac{\sum m_i \mathbf{r}_i}{M}\right) = \frac{\sum m_i \mathbf{v}_i}{M}$$

and subsequently,

$$\mathbf{a}_{CM} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\frac{\sum m_i \mathbf{v}_i}{M}\right) = \frac{\sum m_i \mathbf{a}_i}{M}$$

By recalling that $\sum m_i \mathbf{v}_i$ is the momentum of each point in the system,

$$\mathbf{p}_{\text{net}} = M \cdot \mathbf{v}_{CM}$$
$$\mathbf{F}_{\text{net}} = \frac{d\mathbf{p}}{dt} = M\left(\frac{d\mathbf{v}}{dt}\right) = M\mathbf{a}_{CM}$$

The implication of these equations is that the center of mass of an object moves as if it were a point mass being acted on by some net force. In addition, we can derive the conservation of momentum for a system of particles from these equations: if the net force on a system of particles is zero, the total momentum of the system remains the same.

5 Rotational Physics

Most rotational physics have direct analogues to their linear counterparts, due to the similar derivations of formulas.

5.1 Kinematics

Angular position is defined to be the angle θ of a mass with respect to some reference axis. Even though technically all positions are covered in the domain $[0, 2\pi]$, the angular position of an object is strictly the amount of distance that has been traversed; for example, if a wheel starts at the position $\theta = 0$ and rotates 3 times back to its starting point, its position is $\theta = 6\pi$, not 0. In general, angles are measured in radians due to the nice conversion factor:

$$\text{radian} = \frac{\text{arc length}}{\text{radius}}$$

Note that angles are dimensionless.

$$1 \text{ revolution} = 2\pi \text{ radians} = 360 \text{ degrees}$$

Angular displacement is the change in angular position:

$$\Delta\theta = \theta_f - \theta_0$$

Just as the linear velocity of an object is the time derivative of its position, the angular velocity of an object is the time derivative of its angular position:

$$\omega = \frac{d\theta}{dt}$$

and likewise, angular acceleration is the time derivative of the angular velocity.

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

5.1.1 "Big 5"

The "big 5" equations for linear kinematics can be extended into rotational kinematics.

$$\begin{aligned}
\Delta\theta &= \bar{\omega}t \\
\omega &= \omega_0 + \alpha t \\
\theta &= \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 \\
\theta &= \theta_0 + \omega_f t - \frac{1}{2}\alpha t^2 \\
\omega^2 &= \omega_0^2 + 2\alpha\Delta\theta
\end{aligned}$$

Note that these equations are only valid for uniformly accelerated motion.

5.1.2 Conversions

Due to the use of radians, there are nice conversion factors between the linear and angular counterparts of various quantities.

$$\begin{aligned}
d &= r\theta \\
v &= r\omega \\
a_{\tan} &= r\alpha
\end{aligned}$$

Recall the expression for radial acceleration in a system of uniform circular motion:

$$a_{\text{radial}} = \frac{v^2}{r}$$

This can be converted into an expression of rotational kinematics.

$$\frac{v^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2$$

5.2 Inertia

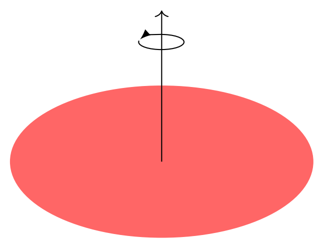
In linear physics, we use mass as a measure of inertia, or the tendency of an object to resist motion. In rotational physics, we extend this concept and use a new quantity, known as the moment of inertia with respect to a specific axis of rotation, equal to

$$I = \int r^2 dm$$

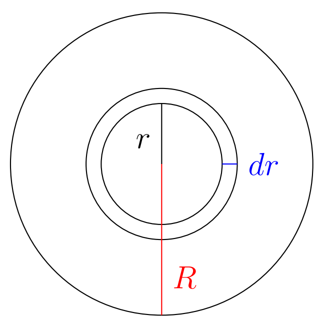
where r is the distance of a little piece of mass from the center of rotation. For point masses, rings, and cylindrical shells rotating about their axes where the radius from their axes does not change, the moment of inertia is simply

$$I = \int r^2 dm = r^2 \int dm = Mr^2$$

In general, however, calculating the moment of inertia can be quite tricky. We need to find an expression for dm in terms of, for instance, r and dr; to do this, we may utilize some simple geometric tricks to find I. For example, take a uniform disc of radius R and mass M:



More clearly,



We take little "chunks" of mass out of the disc in the form of rings, or the locus of points with equal radius from the center. Given that the disc is of uniform density, we know that its surface mass density is constant:

$$\sigma = \frac{M}{A} = \frac{M}{\pi r^2}$$

With some rearrangement,

$$M = \sigma A$$

$$dm = \sigma(dA)$$

$$A = \pi r^2,\ dA = 2\pi r dr$$

$$dm = \sigma(2\pi r)dr$$

We now have our differential piece. Putting it all together,

$$I = \int_0^R r^2(\sigma)(2\pi r)dr = 2\pi\sigma \int_0^R r^3 dr$$

Eventually, this evaluates to

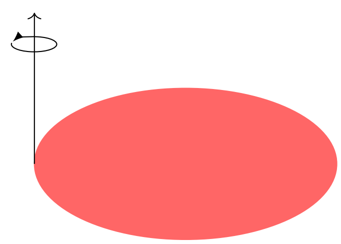
$$\frac{1}{2}MR^2$$

5.2.1 Parallel Axis Theorem

The parallel axis theorem is a nice way to find the moment of inertia of an object about an axis of rotation placed somewhere other than its center of mass. It states,

$$I = I_{cm} + Md^2$$

where d indicates the distance from the new axis of rotation to the center of mass of the object. So, if we were to take our disc from the previous example and instead rotate it about a point on its edge:



we can just use the parallel axis theorem to calculate I about this new axis. We know that $d = R$ and $I_{cm} = \frac{1}{2}MR^2$. Therefore,

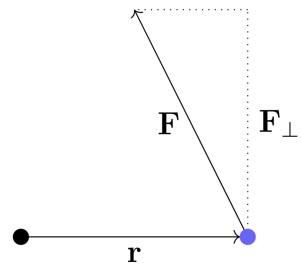
$$I = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$$

5.3 Dynamics

Torque, the angular analogue to force, is the ability of a force to cause an object to accelerate in an angular manner.

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

where **F** is the force applied, and **r** is the relative position vector from the axis of rotation to the point where torque is applied.



The magnitude of torque, then, is

$$|\boldsymbol{\tau}| = |\mathbf{r}||\mathbf{F}|\sin\theta$$

where θ is the angle between the position and force vectors. You'll notice that torque is maximized when the force applied is orthogonal to **r**. Torque has units of newton meters. In computation, it's helpful to note that

$$\tau = rF_\perp = r_\perp F$$

Torque can be defined as a direct analogue of Newton's second law:

$$\tau = I\alpha$$

5.4 Work, Energy, and Power

Analogous to the equation for work in linear mechanics,

$$W = \int \tau d\theta$$

Power is still equal to the change in work over time, or

$$P = \frac{dW}{dt} = \frac{\tau d\theta}{dt} = \tau\omega$$

Following our pattern of similar-looking equations,

$$KE = \frac{1}{2}I\omega^2$$

Recall our relationship between work, kinetic energy, and potential energy:

$$dW = dKE = -dU$$

Since we know that $dW = \tau d\theta$,

$$\tau = \frac{dW}{d\theta} = -\frac{dU}{d\theta}$$

5.5 Momentum

The angular momentum of a point particle is defined to be

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

where $\mathbf{r}$ denotes the relaive position vector from the axis of rotation to the object at hand, and $\mathbf{p}$ denotes its linear momentum. It is often useful to note that the magnitude of this cross product is $|\mathbf{r}||\mathbf{p}|\sin\theta$.

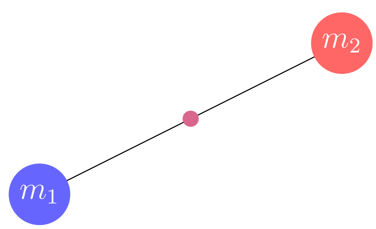
$$|\mathbf{L}| = |\mathbf{r}||\mathbf{p}|\sin\theta = |\mathbf{p}|(|\mathbf{r}|\sin\theta)$$

$|\mathbf{r}|\sin\theta$ is just the perpendicular component of the distance of an object to its axis of rotation. Therefore, we don't need to worry about performing the entire cross product when finding the angular momentum of an object.

Vectors of angular momentum of a system of particles add up to produce a total, "system-wide" vector for angular momentum:

$$\mathbf{L}_{\text{system}} = \sum \mathbf{L}_i$$

For example, a system like a seesaw of some mass M and length ℓ,



will have a total angular momentum of

$$\mathbf{L}_{m_1} + \mathbf{L}_{m_2} + \mathbf{L}_{\text{seesaw}}$$

All objects are moving at the same angular speed, so we can say that

$$\mathbf{L}_{m_1} = I_{m_1}\omega = m_1\left(\frac{\ell}{2}\right)^2\omega$$

$$\mathbf{L}_{m_2} = I_{m_2}\omega = m_2\left(\frac{\ell}{2}\right)^2\omega$$

$$\mathbf{L}_{\text{seesaw}} = I_{\text{seesaw}}\omega = \frac{1}{12}M\ell^2\omega$$

Relationships analogous to linear mechanics are extended to rotational mechanics.

$$\mathbf{L} = I\omega$$

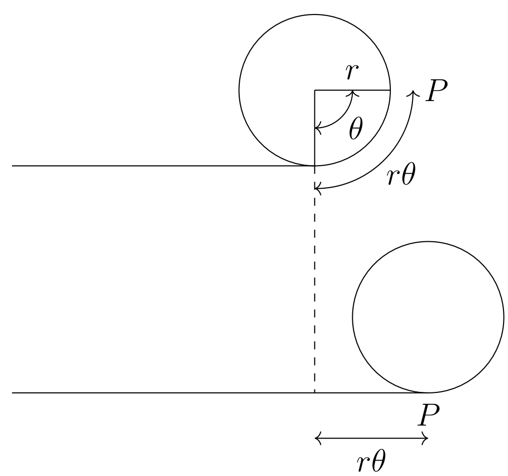
analogous to the linear definition of momentum ($p = mv$). In addition,

$$\boldsymbol{\tau}_{\text{net}} = \frac{d\mathbf{L}}{dt}$$

This leads to an important principle of rotational mechanics: the conservation of angular momentum. If there is no net torque exerted on a system of objects, then the total angular momentum of the system remains constant.

5.6 Rolling without slipping

For physical intuition, we can imagine some spool of thread with a radius of R that is rolling on a surface without slipping, laying down thread.



The translational velocity of the spool is the rate at which the thread is laid down onto the ground, or $\frac{dl}{dt}$. Furthermore, $dl = R|d\theta|$ by definition of radians. Thus,

$$v_{\text{center}} = \frac{dl}{dt} = \frac{Rd\theta}{dt} = R\frac{d\theta}{dt} = R\omega$$

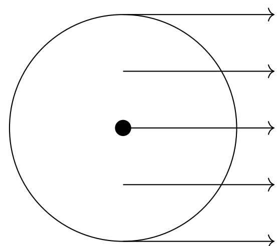
Differentiating this result yields an equation for acceleration:

$$a_{\text{center}} = R\alpha$$

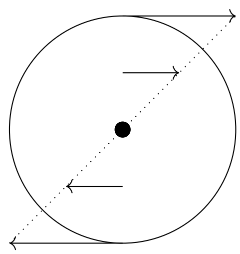
These equations allow us to relate the translational velocity of the *center of mass* of an object with the speed at which it rotates as it moves.

It is also helpful to consider how different components of an object behave under different types of movement.

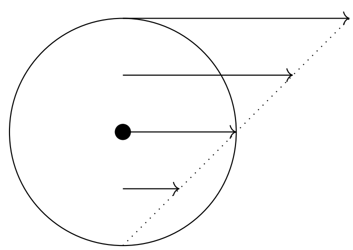
Pure Translation



Pure Rotation



Rotation and Translation



When an object is just being translated, all of its points move at the same velocity as the center of mass. With pure rotation, the speed of the point depends on its proximity to the center of mass via the relationship $v = r\omega$. For a case in which an object rolls without slipping, the velocity of the point in contact with the surface should be 0, and all points move with the translation velocity in addition to their individual rotational velocities.

To calculate the velocity of a single particle in the system, consider that the center of mass is moving at some speed v_{CM}, and that it is a distance

R away from the ground, or the instantaneously stationary axis of rotation. Therefore, the angular speed of the object about that contact point is

$$\omega = \frac{v_{\text{CM}}}{R}$$

We can then apply the formula $v = r\omega$, which yields an expression

$$v = \frac{rv_{\text{CM}}}{R}$$

where r is the distance of the point to the instantaenous axis of rotation, the contact point. We can thus see that the velocity of the top part of some wheel rolling without slipping at some speed v is $2v$, and the velocity of the bottom part is instantaneously 0.

We can calculate the kinetic energy of an object that rolls without slipping:

$$KE = \frac{1}{2}I_{\text{contact}}\omega^2$$

By the parallel axis theorem,

$$I_{\text{contact}} = I_{\text{CM}} + MR^2$$

Substituting this into the original equation,

$$KE = \frac{1}{2}(I_{\text{CM}} + MR^2)\omega^2 = \frac{1}{2}I_{\text{CM}}\omega^2 + \frac{1}{2}MR^2\omega^2$$

$\frac{1}{2}I_{\text{CM}}\omega^2$ is just the rotational kinetic energy of the center of mass about the point of contact. We can transform

$$\frac{1}{2}MR^2\omega^2 = \frac{1}{2}M(R\omega)^2 = \frac{1}{2}Mv_{\text{CM}}^2$$

which is just the translational kinetic energy of the center of mass. Therefore,

$$KE = KE_{\text{pure rotation}} + KE_{\text{pure translation}} = \frac{1}{2}I_{\text{CM}}\omega^2 + \frac{1}{2}Mv_{\text{CM}}^2$$

5.7 Static Equilibrium

In systems without rotation, all it took to state that a system was in equilibrium was that

$$\mathbf{F}_{\text{net}} = 0$$

However, this statement alone is not enough to satisfy the conditions of static equilibrium for extended bodies that may be rotating. A system can be at rest translationally and still rotate, in which case it has non-zero torque. Therefore, to state that a system of extended objects is in equilibrium, we also require the condition

$$\boldsymbol{\tau}_{\text{net}} = 0$$

To solve such a system, we need to satisfy both $\mathbf{F}_{\text{net}} = 0$ and $\boldsymbol{\tau}_{\text{net}} = 0$.

A large part of solving torque problems is strategically choosing a convenient pivot point for the system, perpendicular to the plane on which the forces act. If the net torque about one specific parallel axis is zero, then the torque for all parallel axes is zero. We can use this "trick" to avoid dealing with multiple unknown forces at once.

Any torque exerted by gravity acts as if the gravitational force has been exerted at the center of mass of the object. This is not immediately obvious, but it can be shown through integration.

$$\begin{aligned}
\boldsymbol{\tau} &= \mathbf{r} \times \mathbf{F} \\
d\boldsymbol{\tau} &= \mathbf{r} \times d\mathbf{F} = \mathbf{r} \times \mathbf{g}dm \\
\boldsymbol{\tau} = \int d\boldsymbol{\tau} &= \int \mathbf{r} \times \mathbf{g}dm \\
&= \int \mathbf{r}dm \times \mathbf{g}
\end{aligned}$$

Recall that the center of mass is defined to be

$$\mathbf{r}_{\text{CM}} = \frac{\int \mathbf{r}dm}{M}, \quad \int \mathbf{r}dm = \mathbf{r}_{\text{CM}} \cdot M$$

Substituting into the original equation, then,

$$\boldsymbol{\tau} = M\mathbf{r}_{\text{CM}} \times \mathbf{g}$$

which is just the torque exerted at the center of mass.

6 Simple Harmonic Motion

Simple harmonic motion, often abbreviated SHM, describes any oscillatory motion that has some restoring force or torque with a displacement that can be modeled with sinusoidal functions. At its core, simple harmonic motion indicates any repeating motion where the acceleration is proportional to the displacement of the object.

To start, recall Hooke's law:

$$\mathbf{F}_{\text{net}} = m\mathbf{a} = m\frac{d^2\mathbf{x}}{dt^2} = -k\mathbf{x}$$

From this, we can gather

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Note that this is exactly the relationship stated earlier; the acceleration of the object, or the second derivative of position, is directly proportional by some constant $-\frac{k}{m}$ to the object's displacement. This is a linear second-order differential equation. When solved, it yields two similar sinusoidal functions:

$$x(t) = A\cos\left(t\sqrt{\frac{k}{m}} + \varphi\right)$$
$$x(t) = A\sin\left(t\sqrt{\frac{k}{m}} + \phi\right)$$

with some offsets φ and ϕ. Due to trigonometric identities, we can see that if $\varphi = \phi - \frac{\pi}{2}$, they are equivalent expressions; therefore, the equation we choose is not incredibly relevant when approaching a problem. This is our justification for why sinusoidal functions show up in SHM systems.

We see similar differential equations show up for different kinds of systems, which should serve as a kind of alert. For example, some equation like

$$\tau = I\alpha = I\frac{d^2\theta}{dt^2} - k \cdot \theta$$

we should recognize this as a system of simple harmonic motion. In generality, simple harmonic motion can be expressed as

$$x(t) \text{ or } \theta(t) = A\cos(\omega t + \varphi)$$
$$x(t) \text{ or } \theta(t) = A\sin(\omega t + \phi)$$

for

$$\omega = \sqrt{\frac{k'}{m}} \text{ (linear)}$$
$$\omega = \sqrt{\frac{k'}{I}} \text{ (rotational)}$$

where k' is the effective restoring constant in the differential equation of the form

$$m\frac{d^2x}{dt^2} = -k'x$$

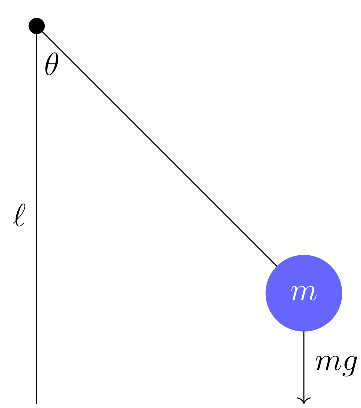
Given equations for ω, we note the relationship between the frequency of the system, the period of the system, and ω.

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

Given these relationships, we can find approximate equations describing period and frequency for both pendulum and spring systems.

6.1 Pendulums

A pendulum undergoes rotational motion, which indicates that we must use τ and θ.



Torque is the perpendicular force times the length of the pendulum, or

$$\tau = -mg\ell \sin\theta$$

The negative sign must exist to indicate that the force is restoring. To find the effective k', we can find the derivative of the force or the torque at the equilibrium point: in this case, we will differentiate torque.

$$k' = -\frac{d\tau}{d\theta}\bigg|_{\theta=0} = -(-mg\ell\cos\theta)\bigg|_{\theta=0} = mg\ell$$

In addition, we know that the moment of inertia of the bob about the rotation point is just $mR^2 = m\ell^2$. Going back to our initial expression for ω,

$$\omega = \sqrt{\frac{k'}{I}} = \sqrt{\frac{mg\ell}{m\ell^2}} = \sqrt{\frac{g}{\ell}}$$

Therefore,

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}$$

and T is just the reciprocal of f. For a physical pendulum,

$$\omega = \sqrt{\frac{mgD}{I}}$$

where D is the distance from the point the object hangs from to its center.

For a torsional pendulum,

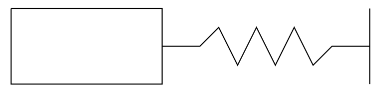
$$\omega = \sqrt{\frac{\kappa}{I}}$$

where κ is the restoring torsional constant.

6.2 Springs

Before we begin, we note the relationship between the position and acceleration equations for a simple harmonic oscillator.

$$\begin{aligned} x(t) &= A\sin(\omega t) \\ v(t) &= A\omega\cos(\omega t) \\ a(t) &= -A\omega^2\sin(\omega t) \\ a(t) &= -\omega^2 x(t) \end{aligned}$$



As noted before, the force due to a spring with constant k stretched or dilated some distance x is

$$F = -kx$$

In addition, we know that the net force on the spring is just

$$F = ma$$

This then means that

$$ma = -kx$$
$$a = -\frac{k}{m}x$$

Our relationship from earlier said that $a = -\omega^2 x$. This therefore means that

$$-\omega^2 = -\frac{k}{m}$$
$$\omega^2 = \frac{k}{m}$$
$$\omega = \sqrt{\frac{k}{m}}$$

and all other quantities can be derived through the relationship

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

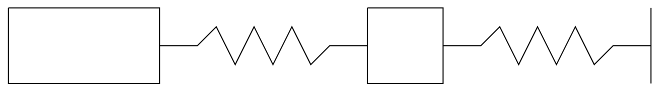
When gravity acts on a spring (for example, one on a diagonal ramp or hanging from a ceiling), only the equilibrium point is shifted, now $\frac{mg}{k}$ away from the end of the spring. This is due to the idea that, when a spring is rotated into a vertical position, the forces now acting on the body at the end of the spring add up to

$$F_{\text{net}} = kx - mg$$

Since we are finding the equilibrium position of the body, we can set F_{net} to 0, and solve algebraically for x to find the new equilibrium point. The period and the oscillation of the vertical spring are the same as if it were a horizontal spring.

6.2.1 Springs in Combination

There are two different ways to configure springs in combination. The first is a system of springs in series, where one spring is placed directly after another:

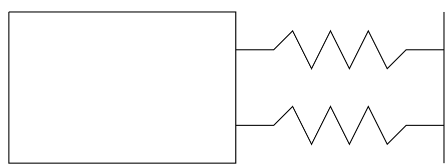


Such a system seems complicated, but we only need to consider k_{eff}; the effective spring constant of the system. For springs in series, we can find this through the relation

$$\frac{1}{k_{\text{eff}}} = \sum \frac{1}{k_n}$$

for an n spring system.

The other way to configure springs in combination is in parallel, working next to each other on the same object.



In a system like this, spring constants add:

$$k_{\text{eff}} = \sum k_n$$

6.3 Energy

For a simple harmonic oscillator, potential energy is equal to

$$U = \frac{1}{2}kx^2$$

and kinetic energy is equal to

$$KE = \frac{1}{2}mv^2$$

Recall our equations for position and velocity:

$$x(t) = A\sin(\omega t)$$
$$v(t) = A\omega\cos(\omega t)$$

Therefore,

$$U = \frac{1}{2}k(A\sin(\omega t))^2 = \frac{1}{2}kA^2\sin^2(\omega t)$$

$$KE = \frac{1}{2}m(A\omega\cos(\omega t))^2 = mA^2\omega^2\cos^2(\omega t)$$

In addition, recall that $\omega^2 = \frac{k}{m}$. Substituting into our equation for KE,

$$KE = \frac{1}{2}mA^2\frac{k}{m}\cos^2(\omega t) = \frac{1}{2}kA^2\cos^2(\omega t)$$

We can then see that

$$U + KE = \frac{1}{2}kA^2(\sin^2(\omega t) + \cos^2(\omega t)) = \frac{1}{2}kA^2$$

The total mechanical energy of a simple harmonic oscillator system will always be $\frac{1}{2}kA^2$. Potential energy is maximized when an object is furthest from the equilibrium point; that is, when it is at the position corresponding to the amplitude. Kinetic energy, on the other hand, is maximized when the object passes through the equilibrium point.

To find the maximum velocity of the object in our system, we consider that this maximum must occur when kinetic energy is maximized; in other words, when it is passing through the equilibrium point. Since the total energy of the system is always constant,

$$\frac{1}{2}mv_{\text{max}}^2 = \frac{1}{2}kA^2$$

7 Universal Gravitation

Newton's universal law of gravitation states that every particle of some mass m_1 experiences a gravitational force to another particle of mass m_2 that is r meters away of size

$$F = G\frac{m_1 m_2}{r^2}$$

where G is the universal gravitational constant, equal to around $6.67 \times 10^{-11}\ \text{N} \cdot \text{m}^2/\text{kg}^2$. If there are more than two masses present in the system, the net force of gravity on one object is the vector sum of all of the gravitational forces exerted on that object by all of the others in the system.

Every object has a gravitational field, or a sphere of influence that surrounds it. This can be visualized as a vector field. The gravitational field for some mass m is equal to

$$g = G\frac{m}{r^2}$$

This g is the acceleration due to the force of gravity from the object of mass m.

The general equation for gravitational potential energy is

$$U = -\frac{GmM}{r}$$

For non-point masses, we must derive the influence of gravity for both areas inside of and outside of the object. The internal effects of gravity for solid and hollow objects behave differently.

Consider a solid sphere of uniform density ρ and a radius R. In addition, consider a small mass m that is a distance r away from the center of the solid sphere.

If $r < R$, we cannot simply use Newton's universal law of gravitation to calculate the magnitude of the gravitational force that the solid sphere exerts on m. Instead, we need to consider only the amount of mass contained within r, since that is the only portion that influences the gravitational force on m. The mass contained within r is just equal to

$$\frac{4}{3}\rho\pi r^3$$

and so the gravitational force exerted by that portion onto m is

$$F_g = G\frac{m\left(\frac{4}{3}\rho\pi r^3\right)}{r^2} = \frac{4}{3}\pi G\rho$$

This equation implies that the force of gravity actually gets stronger as r increases, until m is at the surface of the planet, at which point the force of gravity begins to obey the inverse-square law.

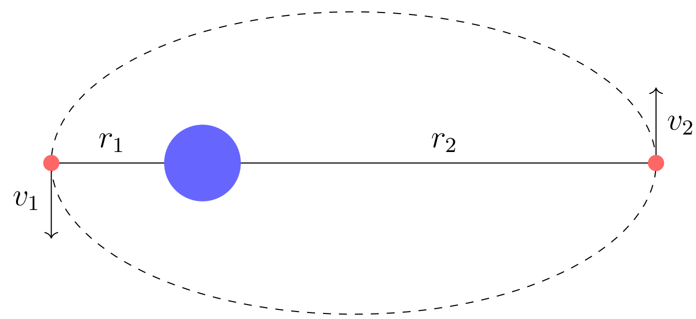
If m is instead inside of a hollow sphere, the gravitational force it experiences is exactly 0.

7.1 Orbits

To find the orbital velocity of a body in a circular orbit, we can equate the force of gravity with the force of centripetal acceleration.

$$
\begin{aligned}
F_g &= F_C \\
\frac{GmM}{r^2} &= \frac{mv^2}{r} \\
v &= \sqrt{\frac{GM}{r}}
\end{aligned}
$$

However, most orbits are elliptical. By Kepler's second law, they obey the law of the conservation of angular momentum.



For some satellite S of mass m (represented by the red circle) orbiting around a planet (represented by the blue circle),

$$
\begin{aligned}
L_1 &= L_2 \\
I_1\omega_1 &= I_2\omega_2 \\
(mr_1^2)\left(\frac{v_1}{r_1}\right) &= (mr_2^2)\left(\frac{v_2}{r_2}\right) \\
r_1 v_1 &= r_2 v_2
\end{aligned}
$$

As an object gets farther away from the the body it orbits, its velocity decreases. Conversely, as an object moves toward the central body, its velocity increases.

Orbits are generally governed by Kepler's laws, which state:

1. The orbits of the planets are ellipses, and the Sun exists at one focus.

2. A planet will sweep out equal areas in its orbits in equal periods of time. This is equivalent to the conservation of angular momentum.

3. The square of the period is proportional to the cube of the semimajor axes. $T^2 \propto R^3$.

More accurately, Kepler's third law states

$$T^2 = \frac{4\pi^2}{GM_{\text{Sun}}} R^3$$

The vis viva equation yields the velocity at any point, given the semimajro axis and the distance from the central body.

$$v^2 = GM \left(\frac{2}{r} - \frac{1}{a} \right)$$

7.2 Escape Velocity

The escape velocity of an object in orbit is the velocity at which it needs to move to completely escape the orbit. To find this escape velocity, we must consider the law of the conservation of energy:

$$U_0 + KE_0 = U_f + KE_f$$

Since we want the object to have neither gravitational potential energy nor kinetic energy with respect to the central body. Substituting in these quantities,

$$-\frac{GmM}{r} + \frac{1}{2} m v_{\text{escape}}^2 = 0$$

With some algebraic manipulations,

$$\begin{aligned} \frac{1}{2} m v_{\text{escape}}^2 &= \frac{GmM}{r} \\ v_{\text{escape}}^2 &= \frac{2GM}{r} \\ v_{\text{escape}} &= \sqrt{\frac{2GM}{r}} \end{aligned}$$