Multivariable Calculus

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1 Vectors and Solid Analytic Geometry

1.1 Vectors

A vector, at its core, is an object that is described by direction and magnitude. In n dimensions, n pieces of information about a point on a plane are required to uniquely describe a vector. Vectors can be described as simple coordinates, which are assumed to have an initial point at the origin and a terminal point at the designated location. Vectors can also be translated from one position to another by way of translating their initial points; these vectors are known as *free vectors*.

The magnitude, or length of a vector is equal to the square root of the sums of the squares of the components, otherwise known as Euclidean distance. For an n-dimensional vector $\mathbf{a}$:

$$|\mathbf{a}| = \sqrt{\sum_{i=1}^{n} a_i^2}$$

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are considered equal if they have both equal magnitudes and directions.

For two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, $\mathbf{P_1P_2} = \langle x_2 - x_1, y_2 - y_1 \rangle$.

1.1.1 Vector Addition

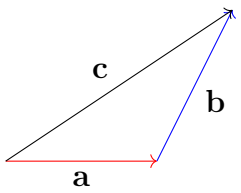
Vectors can be added head-to-tail to form a new vector; corresponding components are added together to form a new coordinate, or the terminal point of the summed vector.

For example, in adding two vectors

$$\mathbf{a} = \langle 2, 0 \rangle$$

$$\mathbf{b} = \langle 1, 2 \rangle$$

The sum, $\mathbf{a} + \mathbf{b} = \mathbf{c}$, looks like



From this triangle, it is apparent that $\mathbf{c} = \langle 3, 2 \rangle$.

Vector addition provides a nice way to express free vectors; a free vector $\mathbf{a}$ that starts at the coordinate point corresponding to the vector $\mathbf{b}$ can simply be described as the sum of the two vectors, or $\mathbf{a} + \mathbf{b}$.

1.1.2 Scalar Multiplication

The operation of multiplying a scalar $c,\ c \in \mathbb{R}$ with a vector $\mathbf{a}$ generates a new vector, each component of which is $c \cdot a_i$. In other words, the constant distributes over all components of a vector. The magnitude of this new vector $c \cdot \mathbf{a}$ is equal to

$$|c| \cdot |\mathbf{a}|$$

and its direction is either the same or direct opposite of the original vector, depending on the sign of the constant; if the constant is negative, then the new vector travels in the exact opposite direction.

Scalar multiplication distributes over addition as well.

$$c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$$

1.1.3 Unit Vectors

Unit vectors are vectors of magnitude 1, most often denoted with a hat ($\hat{a}$). The most elementary of these unit vectors are $\hat{i}$, $\hat{j}$ and $\hat{k}$; these correspond to $\langle 1, 0, 0 \rangle$, $\langle 0, 1, 0 \rangle$, $\langle 0, 0, 1 \rangle$ respectively. From this, it is apparent that any 3-dimensional vector $\mathbf{a}$ can be denoted as a linear combination of these elementary unit vectors:

$$\mathbf{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

This is yet another way of notating a vector.

Any vector can be made into a unit vector; all that needs to be done is to divide the vector by a scalar equal to the magnitude of said vector. With the vector $\mathbf{a}$, its unit vector is

$$\hat{a} = \frac{1}{|\mathbf{a}|}\mathbf{a}$$

1.1.4 Dot Product

The dot product (otherwise known as the scalar or inner product) is the sum of entry-wise multiplication performed on two vectors.

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i \cdot b_i$$

The dot product is associative, commutative, and distributive. The dot product may also be defined in terms of the angle (θ) between the two vectors.

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

The exact proof of this property is derived from the Law of Cosines.

Two angles are orthogonal iff $\mathbf{a} \cdot \mathbf{b} = 0$. This is derived from the angle relationship within the dot product; if the angle between them is $\frac{\pi}{2}$, then $\cos\theta = 0$.

Cauchy-Schwarz Inequality

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$$

This follows from the fact that $|\cos\theta| \leq 1$.

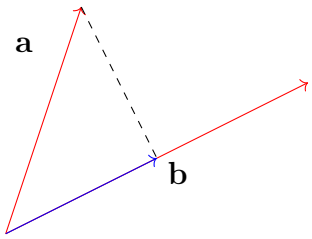
Triangle Inequality

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$$

Similar to the triangle inequality in a geometric context, this inequality simply restates the idea that no side of a triangle can be strictly longer than the sums of the lengths of the other sides (if they were equal to each other, we would have a degenerate triangle, or a line). In this case, the triangle in question is the one formed by two vectors, the side that completes the triangle being the sum.

1.1.5 Component and Projection

Vector projection of $\mathbf{a}$ onto $\mathbf{b}$ is the operation of finding some vector with a length up to $\mathbf{a}$ in the direction of $\mathbf{b}$. This is better explained as a diagram:



The blue vector signifies $\text{proj}_{\mathbf{b}}\mathbf{a}$, read as the "projection of $\mathbf{a}$ onto $\mathbf{b}$". The formula for a vector projection is

$$\text{proj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\cdot\mathbf{b}$$

The magnitude of a projection is the component: $\text{comp}_{\mathbf{b}}\mathbf{a}$.

$$\text{comp}_{\mathbf{b}}\mathbf{a} = \mathbf{a}\cdot\frac{\mathbf{b}}{|\mathbf{b}|}$$

Notice how the projection is simply the component multiplied by the reduced unit vector $\hat{b}$.

1.1.6 Cross Product

The cross product (or the vector product) takes in two vectors and returns another vector. It is convenient to use the determinant method in solving for the cross product.

Given two vectors in $\mathbb{R}^3$, $\mathbf{a}$, $\mathbf{b}$,

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Calculating the determinant of the matrix, we find that

$$\mathbf{a}\times\mathbf{b} = \hat{i}(a_2b_3 - a_3b_2) - \hat{j}(a_1b_3 - a_3b_1) + \hat{k}(a_2b_3 - b_2a_3)$$

The resulting vector is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. This can be shown by computing both $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}$ and $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}$, verifying that they are both equal to 0. The cross product is thus used often when the construction of an orthogonal vector is necessary. Note that the cross product is not commutative with respect to direction; we can use the right hand rule to

determine the order in which we must multiply. We can also simply multiply the vector by -1 if we get an answer of the wrong direction.

The magnitude of a cross product is equal to the magnitudes of the two vectors times the sine of the angle between them.

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

This is, in turn, equal to the area of the parallelogram formed by the two vectors. The volume of a parallelepiped enclosed by three vectors $\mathbf{a},\ \mathbf{b},\ \mathbf{c}$ is $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

1.2 Lines and Planes in Space

1.2.1 Lines

Suppose we have two points, $P_1(x_1,\ y_1,\ z_1)$ and $P_2(x_2,\ y_2,\ z_2)$. The vector $\mathbf{a}$ that connects P_1P_2 is equal to

$$\langle x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1 \rangle$$

However, for now, $\mathbf{a}$ is assumed to have a starting point at the origin, and travel for a finite distance; to fix this, we may perform vector addition and multiplication:

$$\langle x,\ y,\ z \rangle = \langle x_1,\ y_1,\ z_1 \rangle + t\langle a_1,\ a_2,\ a_3 \rangle$$

This is the vector form of a line. Notice that each component among $x,\ y,\ z$ correspond to different equations; we can break the vector line into the equations

$$\begin{aligned} x &= x_1 + a_1 t \\ y &= y_1 + a_2 t \\ z &= z_1 + a_3 t \end{aligned}$$

This is the form of a parametric equation, where t is our parameter; the form is, therefore, called parametric form. We can take this one step further; in all three equations, we may isolate t:

$$\begin{aligned} t &= \frac{x - x_1}{a_1} \\ t &= \frac{y - y_1}{a_2} \\ t &= \frac{z - z_1}{a_3} \end{aligned}$$

It is then valid to state that

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{a_2} = \frac{z - z_1}{a_3}$$

This is the symmetric form of a line equation.

Two lines are parallel if their direction vectors ($\mathbf{a}$) are scalar multiples of each other; they will obviously not intersect. However, there exist skew lines; lines that will never intersect, but are nonetheless not parallel. To determine if and where two lines ℓ_1 and ℓ_2 intersect, we can set different equations in parametric form equal to each other. If there exists a single t that fulfills conditions for all three systems, the lines intersect at time t; else, they do not intersect.

1.2.2 Planes

A plane can be described by two pieces of information; a normal vector and a point. Given a point $P(x_1, y_1, z_1)$ and a normal (orthogonal) vector $\langle a_1, a_2, a_3 \rangle$, the equation of a plane containing the point P orthogonal to $\mathbf{a}$ is

$$a_1(x - x_1) + a_2(y - y_1) + a_3(z - z_1) = 0$$

If we have three points $P,\ Q,\ R$, we can find the plane on which all three of them lie by taking the cross product $\mathbf{PQ} \times \mathbf{PR}$ and then using one of the points to find the equation of the plane.

Two planes are parallel if their normal vectors are parallel; similarly, two planes are orthogonal if their normal vectors are orthogonal.

Intersecting Planes

We can find lines as intersections of two planes. Given

$$\mathcal{P}_1 : a_1x + a_2y + a_3z + d_1 = 0$$

$$\mathcal{P}_2 : b_1x + b_2y + b_3z + d_2 = 0$$

By setting one of the variables ($x,\ y,\ z$) to zero, we get a solvable linear system of equations, out of which we can find a common point between the two planes. We may also take the cross product $\mathbf{a} \times \mathbf{b}$ to find the direction vector $\mathbf{c}$ (every line on a plane is orthogonal to the plane's normal vector; therefore, a line intersecting two planes must be orthogonal to both of their direction vectors). We thus have all of the information necessary to find a line intersecting the planes; a point and a direction vector.

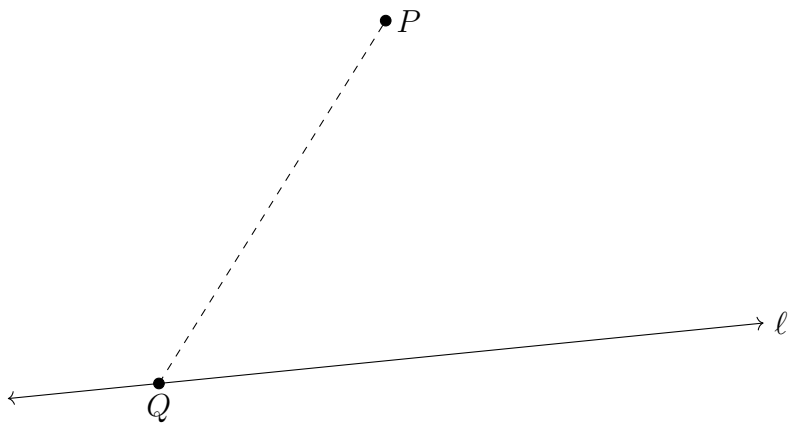
1.2.3 Distance

There are six cases of distance between different entities that we must calculate.

Point - Point
For points $P_1(x_1,\ y_1,\ z_1)$, $P_2(x_2,\ y_2,\ z_2)$, we can just use the Euclidean distance formula:

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Point - Line
For a point $P(x_1,\ y_1,\ z_1)$ and a line ℓ defined by a point $Q(x_2, y_2, z_2)$ and a direction vector $\mathbf{a}$, we can construct a diagram:



We can project the vector **QP** onto the direction vector **a**; this would give us one leg of a right triangle. To get the actual perpendicular distance, we would need to apply the Pythagorean theorem using **QP** as our hypotenuse and our projection as our horizontal leg.
Point - Plane
We can construct a vector from the point $P(x_1,\ y_1,\ z_1)$ to any point on the plane in question, $Q(x_2, y_2, z_2)$. Then, we simply need to take the component of **PR** to the normal vector of the plane **n**.
Line - Line
If the lines intersect in any way, then the distance is simply 0. If the lines are parallel, we can choose any point on one of the lines and perform a point

- line distance calculation. If the lines are skewed with respect to each other, then

1. Find a plane $\mathcal{P}$ such that $\mathcal{P}$ is contains one of the lines and is parallel to the other. We can find the normal vector of this plane by taking the cross product of the direction vectors of the two lines.

2. Then, use the Line - Plane distance calculation.

Line - Plane
If the line is not parallel to the plane, then the distance is 0. If it is parallel to the plane, then we can reason that every single point on the line will be equidistant from the plane; we can simply use the Point - Plane calculation by finding some point on the plane.
Plane - Plane
If the planes are not parallel, then the distance is 0. If the planes are parallel, then we can use the Point - Plane calculation after finidng a point on one of the planes.

1.3 Surfaces

When graphing more complicated three-dimensional surfaces, it is often helpful to sketch "traces" of the graph on the different coordinate planes. To do this, we set one of our variables to zero at any given time and draw a trace of what the graph looks like on the particular coordinate plane. With three traces, we may interpolate between the sketches and draw guiding lines in order to arrive at a full sketch or graph.

1.3.1 Cylinders

If we have some curve $\mathcal{C}$ and a line ℓ not parallel to or inside of the plane of the curve, a cylinder is the locus of all points on the lines parallel to ℓ that intersect $\mathcal{C}$. $\mathcal{C}$ is called the directrix of the cylinder, and ℓ is called the ruling of the cylinder. The equations of such figures usually describe a 2-dimensional shape in one particular plane, leaving out a third variable. The equation $y = x^2$, for example, describes a parabola in the $x - y$ plane, which is extended in the z-direction.

1.3.2 Surfaces of Revolution

A surface of revolution is a generated by a revolving a plane curve $\mathcal{C}$ about a line, most commonly one of the coordinate axes.

1.3.3 Quadric Surfaces

Extended from two-dimensional conic graphs, we may derive six surfaces and three equations describing those surfaces. The equations take the form of

$$ax^2 + by^2 + cz^2 = 0 \tag{1}$$
$$ax^2 + by^2 + cz^2 = 1 \tag{2}$$
$$ax = by^2 + cz^2 \tag{3}$$

Of course, variables may be switched around; this just yields the same surface, just created from a different perspective. There are six surfaces created from these equations: Cones, Ellipsoids, Hyperboloids (of one sheet or two sheets), Paraboloids, and Hyperbolic Paraboloids.

Cone

A cone occurs whenever one of the coefficients of equation (1) is negative; for example, $x^2 + y^2 - z^2 = 0$ describes a cone. Two coefficients being negative yields what is essentially the same equation, since we may multiply the entire equation by a negative scalar. All of the coefficients cannot be positive at once; three positive numbers cannot add up to zero, excluding the case where $(x, y, z) = (0, 0, 0)$; this is the degenerate case.

Ellipsoid

An ellipsoid takes the form of equation (2) when all coefficients are positive. This graph can be derived from the traces of the surface on each of the coordinate planes, which turn out to be ellipses.

Hyperboloid of 1 sheet

A hyperboloid of 1 sheet occurs with equation (2) when one of its coefficients is negative.

Hyperboloid of 2 sheets

A hyperboloid of 2 sheets occurs with equation (2) when two coefficients are negative.

Paraboloid

A paraboloid occurs with equation (3) when all coefficients are positive.

Hyperbolic Paraboloid

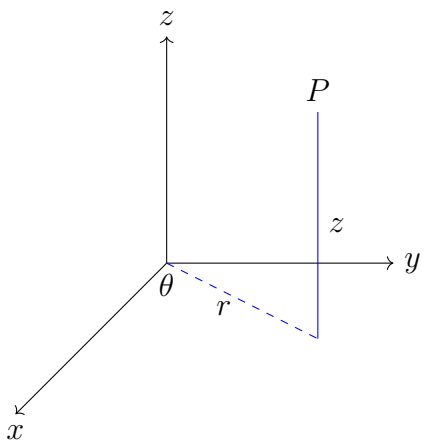
A hyperbolic paraboloid occurs with equation (3) when one of the constants in front of the squared term is negative.

2 Polar Coordinates in Three Dimensions

2.1 Cylindrical Coordinates

Cylindrical coordinates represent a single point with a tuple (r, θ, z). It can be thought of as an elevation of the 2-dimensional polar coordinate system, with a rectangular z coordinate to represent height.

1. r is the radius of the polar coordinate for the projection of a point P onto the xy-plane.
2. θ is the angle between the x-axis and the direction of OP', where P' is the projection of P onto the xy-plane.
3. z is the elevation of P', acting as a rectangular coordinate.



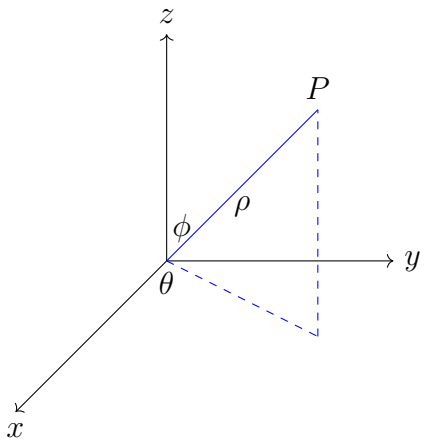
Rectangular and cylindrical coordinates have the relations:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \\ \tan\theta &= \frac{y}{x} \\ r^2 &= x^2 + y^2 \end{aligned}$$

2.2 Spherical Coordinates

In spherical coordinates, a point is represented by the tuple (ρ, θ, ϕ) where

1. ρ is distance between the origin and the point $|\overline{OP}|$
2. θ is the angle between the x-axis and $\overline{OP'}$ (where P' is the projection of P onto the xy-plane)
3. ϕ is the angle betweeen the positive z-axis and $\overline{OP}$



We have the relations:

$$\begin{aligned}
\rho^2 &= x^2 + y^2 + z^2 \\
r &= \rho \sin\phi \\
x &= r\cos\theta = \rho\sin\phi\cos\theta \\
y &= r\sin\theta = \rho\sin\phi\sin\theta \\
z &= \rho\cos\theta
\end{aligned}$$

3 Vector Valued Functions

3.1 Definitions

A vector-valued function takes in some parameter t and converts it into a vector.

$$\mathbf{r}(t) : \mathbb{R} \to \mathbb{R}^n$$

where n indicates the number of dimensions, which is commonly 3. These functions look something like

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$$

3.1.1 Curves

A vector-valued function can represent some curve in space.

$$\mathcal{C} = \{(x(t), y(t), z(t)) \mid t \in \mathbb{R}\}$$

A curve is smooth within an interval if none of its derivatives ever reach 0 simultaneously. That is, there is no t within an interval (a, b) within which $f'(t) = g'(t) = h'(t) = 0$.
A curve is considered closed if, for two points $a,\ b$, $\overline{r}(a) = \overline{r}(b)$.

3.1.2 Arclength

The arclength of some smooth curve $\mathcal{C} = (f(t), g(t), h(t))$ in the interval $[a, b]$ is

$$\ell = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2} dt$$

Note that this is the integral of the magnitude of the vector $\langle x',\ y',\ z' \rangle$.

3.2 Limits, Derivatives, Integrals

3.2.1 Limits

$$\lim_{t \to a} \mathbf{r}(t) = \left[\lim_{t \to a} f(t)\right] + \left[\lim_{t \to a} g(t)\right] + \left[\lim_{t \to a} h(t)\right]$$

From this information, it is necessarily the case that the derivative and integral of $\mathbf{r}(t)$ are performed component-wise, as they depend on summation (which is a component wise operation) and the limit (also a component wise operation). Limit properties from single variable calculus carry over:

$$\begin{aligned}
\lim_{t \to a}[\mathbf{u}(t) + \mathbf{v}(t)] &= \lim_{t \to a} \mathbf{u}(t) + \lim_{t \to a} \mathbf{v}(t) \\
\lim_{t \to a}[\mathbf{u}(t) \times \mathbf{v}(t)] &= \lim_{t \to a} \mathbf{u}(t) \times \lim_{t \to a} \mathbf{v}(t) \\
\lim_{t \to a}[\mathbf{u}(t) \cdot \mathbf{v}(t)] &= \lim_{t \to a} \mathbf{u}(t) \cdot \lim_{t \to a} \mathbf{v}(t)
\end{aligned}$$

3.2.2 Differentiation

Many of the same properties for differentiation in single variable calculus carry over to vector-valued functions.

$$\begin{aligned}
\frac{d}{dt}(\mathbf{u}(t)+\mathbf{v}(t)) &= \mathbf{u}'(t)+\mathbf{v}'(t) \\
\frac{d}{dt}(c\mathbf{u}(t)) &= c\mathbf{u}'(t) \\
\frac{d}{dt}(\mathbf{u}(t)\cdot\mathbf{v}(t)) &= \mathbf{u}'(t)\cdot\mathbf{v}(t)+\mathbf{u}(t)\cdot\mathbf{v}'(t) \\
\frac{d}{dt}(\mathbf{u}(t)\times\mathbf{v}t) &= \mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)
\end{aligned}$$

3.2.3 Integration

Both parts of the Fundamental Theorem of Calculus apply to vector-valued functions. We know that

$$\int \mathbf{u}(t)dt = \left\langle \int f(t)dt,\ \int g(t)dt,\ \int h(t)dt \right\rangle$$

Properties for integration are extended to vector valued functions.

3.3 Unit Tangent and Normal

For a vector valued function $\mathbf{r}(t)$, there exists both a unit normal and unit tangent vector for any point at an arbitrary time t.

The unit tangent vector travels exactly one unit in the tangent direction of a curve at a point, while the unit normal vector travels exactly one unit in a normal direction to the curve (and the tangent) at a given point.

Therefore, the unit tangent is given by

$$\mathbf{T}(t) = \frac{1}{|\mathbf{r}'(t)|}\mathbf{r}'(t)$$

and the unit normal is given by

$$\mathbf{N}(t) = \frac{1}{|\mathbf{T}'(t)|}\mathbf{T}'(t)$$

3.4 Curvature

The curvature K of some curve is defined to be the change in the angle between the unit vector $\mathbf{i}$ and the tangent vector at the point $\mathbf{T}(s)$, where s is a parameter corresponding to the arclength. In other words,

$$K = \left| \frac{d\theta}{ds} \right|$$

The curvature of a circle with a radius a is always a constant $\frac{1}{a}$. Intuitively, this makes sense because a circle has less curvature at a given point if its radius is larger, and vice versa.

We know that

$$s(x) = \int_a^x \sqrt{1 + (y')^2} dx$$

$$\theta = \arctan y'$$

and we know, through the chain rule, that

$$\frac{d\theta}{dx} = \frac{d\theta}{ds} \frac{ds}{dx}$$

Consquently, the curvature of a graph of a single-variable function is

$$K = \left| \frac{y''}{(1 + (y')^2)^{\frac{3}{2}}} \right|$$

Similarly, for a curve defined with two functions such that $x = f(t)$ and $y = g(t)$,

$$K = \left| \frac{f'g'' - g'f''}{((f')^2 + (g')^2)^{\frac{3}{2}}} \right|$$

K is also defined as

$$K = |\mathbf{T}'(s)|$$

where s is the arclength parameter.

3.5 Acceleration

Some reminders:

1. Speed $= |\mathbf{r}'| = |\mathbf{v}| = v = \frac{ds}{dt}$

2. $\mathbf{T}(t) = \frac{\mathbf{r}(t)'}{|\mathbf{r}'(t)|}$, and $\mathbf{T}(t) = \mathbf{T}(s)$ because the tangent is invariant to parameterization.
3. $K := \left|\frac{d}{ds}\mathbf{T}\right|$
4. $K \cdot \mathbf{N}(s) = \frac{d}{ds}\mathbf{T}$
5. $K = \frac{1}{\rho}$

The acceleration, or the second derivative of $\mathbf{r}(t)$ can be broken up into tangential and normal parts.

$$\mathbf{r}''(t) = \mathbf{a} = \frac{dv}{dt}\mathbf{T} + \frac{v^2}{\rho}\mathbf{N}$$

where ρ is the radius of curvature at a point. Essentially, this equation splits up the acceleration into "speeding up" and "turning" parts. For brevity, we may state

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$$

where a_T and a_N are the tangential and normal parts of the acceleration, respectively.

We can derive computable equations for a_T and a_N.

$$a_T = \frac{\mathbf{r}' \cdot \mathbf{r}''}{|\mathbf{r}'|}$$

$$a_N = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|}$$

We can also derive that

$$|\mathbf{a}|^2 = a_T^2 + a_N^2$$

This equation, intuitively, makes sense: a_T and a_N form the horizontal and vertical parts of the acceleration at some instant, and therefore, the actual acceleration would arise from the pythagorean theorem with these components. In addition, we can derive an equation for K for three dimensions:

$$K = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$$

4 Derivatives of Multivariable Functions

Multivariable functions are, in a way, the opposite of vector valued functions; they take in multiple independent values and return a single value as an output. Formally, a multivariable f exists such that

$$f : D \to R, \quad D \subseteq \mathbb{R}^n, \quad R \subseteq \mathbb{R}$$

There are several ways to sketch these functions. We may calculate traces of the graph in some space and connect the spaces between them to generate some image, or, we may use level curves.

A level curve is a locus of points with a fixed output value. They are sketched as multiple curves on a single surface, much like a topographical map. To sketch a level curve of a function $f(x, y)$, we can set the function to some function k and compute equations for the graph in a single-variable context for each different k.

4.1 Limits and continuity

As everything in single variable calculus depends on limits, everything in multivariable calculus also depends on limits. In multivariable calculus, a limit is visualized as any path (curve) moving towards a specific point on a domain space that may correspond to a value on the output space. In single variable calculus, limits are formally defined with epsilon delta proofs; the same idea exists in multivariable calculus, but the "distance" of δ away from the point is instead notated as B_δ (a ball of radius δ).

Let a multivariable function f be defined near some point (a, b), but possibly not at the point itself. The limit as $f(x, y)$ approaches (a, b), is L

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every $\epsilon > 0$ there is a corresponding $\delta > 0$ such that if

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

then

$$|f(x, y) - L| < \epsilon$$

In calculating these limits, we may first plug the points in. If that does not yield an answer, we can "choose" curves to travel along in order to find

the limit. For example, we may choose a curve such that y is always 0 and x is the changing value, turning our operation into a single-variable limit. However, caution is necessary; if two different curves to a point produce different limiting values, then the limit does not exist for that function at that point.

4.2 Partial Derivatives

Recall that the definition of a derivative in single variable calculus is stated as

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

This same notion is carried over into a multivariable system.

$$\frac{\partial f}{\partial x} = f_x = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$\frac{\partial f}{\partial y} = f_y = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

This definition extends to multiple dimensions; we are not limited to just two.

Note that when we take a partial derivative with respect to x, we are holding the y aspect fixed; the same goes for a partial derivative with respect to y. Thus, it is fair to say that when we take a partial derivative of some variable, we hold all of the other variabes constant; we may treat them as such when computing these derivatives. Taking a partial derivative works with the same techniques as it does in single calculus, like the product and quotient rules.

Taking higher derivatives works the same way, with the added bonus that we can take higher derivatives with respect to different variables. Since a derivative is a function-to-function operation, we simply take another derivative of our resulting expression.

4.2.1 Clairaut's theorem

Clairaut's theorem states that, for a multivariable function $f(x, y)$, if f and all of its first and second order derivatives are continuous on some region R, then

$$f_{xy} = f_{yx}$$

on R.

4.3 Differentials and Increments

4.3.1 Increments

For some $w = f(x, y)$, let Δx and Δy be the increments of x and y, respectively. Then,

$$\Delta w = f(x + \Delta x, y + \Delta y) - f(x, y)$$

Effectively, Δw is the change in w if its inputs are shifted by some $\Delta x,\ \Delta y$. Now, calculating Δw is challenging for a multitude of reasons, especially if done by hand. Thus, increments are approximated by differentials.

We can first think about increments from a single variable perspective. Let

$$\Delta u = f(x_0 + \Delta x) - f(x_0)$$

This looks similar to the definition of a single variable derivative, so we can divide by Δx on both sides and take a limit:

$$\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = f'(x_0)$$

$$\lim_{\Delta x \to 0} \left[\frac{\Delta u}{\Delta x} - f'(x_0) \right] = 0$$

We can abstract the inner contents of this limit away; call it some ϵ.

$$\epsilon = \frac{\Delta u}{\Delta x} - f'(x_0)$$

Thus,

$$\Delta u = f'(x_0)\Delta x + \epsilon \Delta x$$

This is an alternate definition for an increment in a single variable context. There is no reason why the derivation of this definition should be limited to a single variable system; we can perform a similar process, but with two variables instead, and we arrive at the definition

$$\Delta w = f_x(x_0, y_0)\Delta x + \epsilon_1 \Delta x + f_y(x_0, y_0)\Delta y + \epsilon_2 \Delta y$$

where ϵ_1 and ϵ_2 are simply extensions of the singular ϵ definition we had earlier. Keep in mind that these ϵs are functions of $\Delta x,\ \Delta y$ and have a limit of 0 as $(\Delta x,\ \Delta y) \to (0,\ 0)$.

4.3.2 Differentials

In single variable calculus, we defined the differential of a function y as

$$dy = f'(x)dx$$

where dx is equal to the increment Δx, and dy is approximately equal to Δy for small values. In multivariable calculus, the differential of a function w is

$$dw = f_x(x, y)dx + f_y(x, y)dy$$

where, again, $dx = \Delta x$ and $dy = \Delta y$.

There are several useful statements about continuity that fall out of this definition.

If $w = f(x, y)$, then f is differentiable at (x_0, y_0) if it can be expressed in our incremental form from earlier,

$$\Delta w = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$$

If $w = f(x, y)$, and f_x and f_y are continuous on a rectangular region R, then f is differentiable on R.

If f is differentiable at (x_0, y_0), then f is continuous at x_0, y_0.

The same definition of differentials extends to three dimensions:

$$df = f_x dx + f_y dy + f_z dz$$

4.4 Chain Rule

In single variable calculus, the chain rule states that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or simply

$$y(u(x)) = y'(u(x))u'(x)$$

In multivariable calculus, for some function $w = f(u, v)$ where $u = g(x, y)$ and $v = k(x, y)$,

$$w_x = w_u \cdot u_x + w_v \cdot v_x$$

$$w_y = w_u \cdot u_y + w_v \cdot v_y$$

or

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial x}$$
$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial y}$$

This can be extended to multiple dimensions: for some w that is made up of n variables $u_1, u_2, \cdots u_n$ that are, in turn, each made up of m variables $x_1, x_2, \cdots x_m$ with continuous first partial derivatives, then

$$\frac{\partial w}{\partial x_j} = \sum_{i=1}^{n} \frac{\partial w}{\partial u_i} \cdot \frac{\partial u_i}{\partial x_j}$$

4.4.1 Implicit Differentiation

Implicit differentiation, like many other things from single variable calculus, makes an appearance in multivariable calculus.

If some function $F(x, y) = 0$ implicitly defines a function of one variable, $y = f(x)$, then

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}$$

Similarly, if some function $F(x, y, z) = 0$ implicitly defines a function of two variables $z = f(x, y)$, then

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}$$

4.5 Directional Derivatives

Instead of thinking of partial derivatives as operations along our coordinate axes, we can think of them as operations along the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and so forth. If this is so, then there is no reason we shouldn't be able to take derivatives along other paths, or vectors, of direction. The directional derivative of a function f along some unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is defined as

$$D_{\mathbf{u}} f(x, y) = \lim_{t \to 0} \frac{f(x + tu_1, y + tu_2) - f(x, y)}{t}$$

This limit seems like a pain to compute. So, we can do some simplification. Define some function $g(t)$ such that

$$g(t) = f(x + tu_1, y + tu_2)$$

Recall, also, the alternate definition of a derivative that specifies a point.

$$g'(0) = \lim_{t \to 0} \frac{g(t) - g(0)}{t - 0}$$

It follows that

$$g'(0) = \lim_{t \to 0} \frac{f(x + tu_1, y + tu_2) - f(x, y)}{t} = D_{\mathbf{u}} f(x, y)$$

We can also write $g(t)$ as a composition of different functions, say, $r,\ v$, where

$$r = x + tu_1,\ \ v = y + tu_2$$

From the chain rule, this becomes

$$g'(t) = f_r(r, v)\frac{\mathrm{du}}{\mathrm{dt}} + f_v(r, v)\frac{\mathrm{dv}}{\mathrm{dt}}$$

which is equal to

$$f_r(r, v)u_1 + f_v(r, v)u_2$$

Recall that the directional derivative is equal to the above equation evaluated at 0: then, $r = x$ and $v = y$, and the formula becomes

$$f_x(x, y)u_1 + f_y(x, y)u_2$$

Therefore, the direction derivative of a function along the unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}} f = f_x u_1 + f_y u_2$$

Note that this is equal to taking a dot product:

$$\langle f_x, f_y \rangle \cdot \mathbf{u}$$

We'll introduce a new operation, ∇. This symbol, the nabla (pronounced 'del') takes the gradient of the function f.

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle$$

$$\nabla f(x, y) = \langle f_x, f_y \rangle$$

It follows that

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$$

The maximum value of $D_{\mathbf{u}} f$ at some point P occurs in the direction of ∇f evaluated at P; this maximum value is $|\nabla f(x, y)|$.

Similarly, the minimum value of $D_{\mathbf{u}} f$ (or the 'most negative' value) occurs in the opposite direction of ∇f, or in the direction of $-\nabla f$. This maximum value is also $|\nabla f(x, y)|$.

Traveling on along the path orthogonal to ∇f will yield an unchanging value of $D_{\mathbf{u}} f$.

4.6 Tangent Planes

Given a surface, S (defined by some function $F(x, y, z) = 0$), and a point on that surface, P_0, there exists a line normal to S at P_0 (given that S is not just a singular point). Since a line defines a unique plane perpendicular to that line, the plane defined by the normal line is tangent to S at P_0.

Given that $F(x, y, z)$ has continuous first partials and that P_0 is a point on the surface defined by F, then the vector $\nabla F|_{P_0}$ is normal to S at P_0.

In equation form, this line is

$$\mathbf{r_0} + t \cdot \nabla F|_{P_0}$$

Even further, the equation for a tangent plane to S at $P_0(x_0, y_0, z_0)$ is

$$\nabla F|_{P_0} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

or

$$\nabla F|_{P_0} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$$

An important thing to note is that ∇F cannot be $\mathbf{0}$.

4.7 Extrema

Analogous to single variable critical points, multivariable critical points are locations in which the components of ∇F are all either 0 or do not exist. Local extrema must occur at these critical points, but critical points do not necessarily have to be extrema. On the other hand, absolute extrema do

not necessarily have to occur at these critical points; they can occur on the boundaries of their domains as well.

To solve problems where the requirement is to find absolute extrema on some rectangular domain, we can construct a collection of possible locations of extrema; the first points in the list should be critical points and points on the vertices of the domain. We can, in addition, limit the multivariable function to some boundary curve, which turns the function into a single variable function that can be optimized with single variable strategies.

4.7.1 Classifying Extrema

Given that f has continuous second partial derivatives on some rectangular region Q, define a function $g(x, y)$ such that

$$g(x, y) = f_{xx}f_{yy} - [f_{xy}]^2$$

for all pairs (x, y) in Q. For some critical point (a, b) in Q, then

1. (a, b) is a local maximum if $g(a, b) > 0$ and $f_{xx}(a, b) < 0$.
2. (a, b) is a local minimum if $g(a, b) > 0$ and $f_{xx}(a, b) > 0$.
3. (a, b) is a saddle point if $g(a, b) < 0$.
4. (a, b) is an inconclusive point if $g(a, b) = 0$.

4.8 Lagrange Multipliers

Lagrange multipliers are a more efficient way to solve optimization problems, in which we are tasked with finding the optimum of some equation subject to a constraint.

For a function f and a constraint equation g, if f has an extremum at (x_0, y_0) and $\nabla g(x_0, y_0) \neq \mathbf{0}$, then there exists some number λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

This number, λ, is known as the Lagrange multiplier.

Out of this method, we receive a system of three equations, for which we can solve for three variables: x, y, and λ.

1. $f_x(x, y) = \lambda g_x(x, y)$

2. $f_y(x, y) = \lambda g_y(x, y)$

3. $g(x, y) = C$

5 Double Integration

In single-variable calculus, the formal definition of a definite integral was the sum of partitions as the limit of the largest partition size approaches zero.

$$\lim_{||\Delta|| \to 0} \sum_i f(w_i) \Delta x_i$$

In a space of two variables, integration follows the same principle of dividing up the area of the domain.

There are two kinds of regions that can be considered:

1. **Type I** regions are bound between two constants in the x direction and two functions of x, $g_1(x)$ and $g_2(x)$.

2. **Type II** regions are bound between two constants in the y direction and two functions of y, $h_1(y)$ and $h_2(y)$.

Any region R can be decomposed into regions of one or the other type. The region can be divided into little rectangles of area with something like a grid. Then, the totality of these closed rectangles that all reside entirely within R are called the inner partition of R.

The analogue of a Riemann sum in two dimensions is, then, the product of the components of the inner partition of R with the value of f at some location within the rectangle.

$$\sum_{i=1}^{n} f(u_i, v_i) \Delta A_i$$

where R_i contains (u_i, v_i) and ΔA_i is the area of R_i.

We introduce a new symbol known as the double integral of f over R, denoted:

$$\iint_R f(x, y) dA$$

This is defined to be the limit as the norm of the largest such partition R_i approaches 0:

$$\iint_R f(x,y)dA = \lim_{||P||\to 0} \sum_i f(u_i, v_i)\Delta A_i$$

If the double integral of f over R exists, then f is integrable over R. The volume V of a solid that lies under a function $z = f(x,y)$ over the region R is, in addition, equal to the double integral:

$$V = \iint f(x,y)dA$$

Double integrals have the following properties:

$$\iint_R cf(x,y)dA = c\iint_R f(x,y)dA \tag{4}$$

$$\iint_R [f(x,y) + g(x,y)]dA = \iint_R f(x,y)dA + \iint_R g(x,y)dA \tag{5}$$

$$\iint_R f(x,y)dA = \iint_{R_1} f(x,y)dA + \iint_{R_2} f(x,y)dA \tag{6}$$

$$\iint_R f(x,y)dA \geq 0 \tag{7}$$

Equation (6) is only valid if R is the union of the nonoverlapping regions R_1 and R_2. Equation (7) is only valid if $f(x,y) \geq 0$ throughout R.

However, you'll notice that evaluating these double integrals directly is almost impossible, which is why we require the use of other methods for the actual computation.

5.1 Iterated Integrals

Just like we have partial differentiation, we have a concept known as partial integration. For example, consider the expression

$$\int_a^b f(x,y)dy$$

This simply means that, while x is kept constant, we are integrating with respect to y between the bounds of a and b. Therefore, to find the area of a

region in a 2-D space, we can place one 'partial integral' inside another:

$$\int_a^b \int_c^d f(x,y)dydx$$

In this case, we would evaluate the inner integral first, and then evaluate the resulting expression with respect to the outer integral — this approach is known as iterated integration.

In addition, we can deal with regions of types I and II with iterated integration. For a type I region,

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

would give the volume of the solid defined by f and the type I region. Notice how the inner integral is computed with respect to y, because the bounds on the integral are functions of x. Because we expect to get a constant out of our evaluation (in calcuating volume), the function bounds cannot be in the outer integral. Similarly,

$$\int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)dxdy$$

would yield the volume of the solid defined by f over a type II region.

Iterated integrals make the computation of double integrals much, much easier:

$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

for a region of type I and

$$\iint_R f(x,y)dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)dxdy$$

for a region of type II.

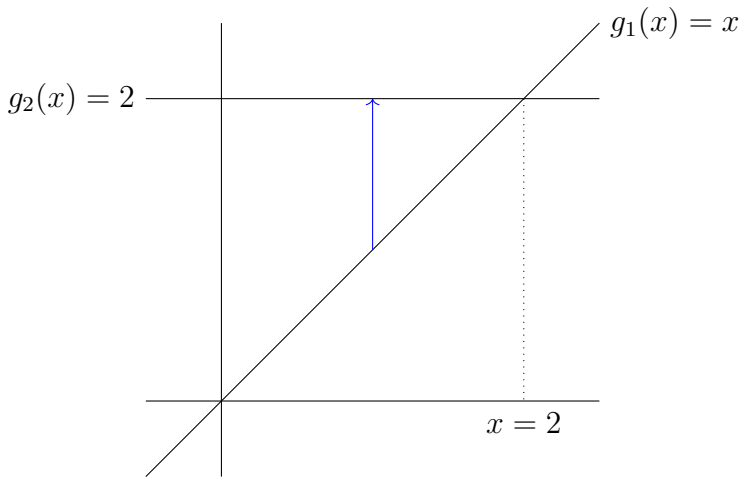
5.1.1 Fubini's Theorem

Some double integrals cannot be solved in the order of integration they are given in; sometimes, switching the order of integration is required. Fubini's

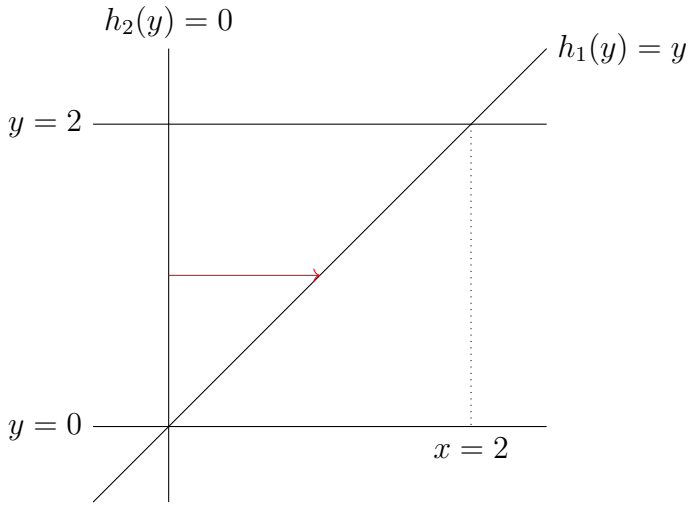
theorem, which states that double integrals can be expressed as iterated integrals, implies that the order of integration can thus be switched if a region R of a specific type can be converted into the other. For example,

$$\int_0^2 \int_x^2 e^{y^2} dy dx$$

cannot be computed, as the antiderivative of e^{y^2} does not have a closed form answer. However, we can draw out the region itself:



We notice that this region of type I can be expressed as a region of type II instead:



This second image corresponds with the iterated integral

$$\int_0^2 \int_0^y e^{y^2} dx dy$$

which we can evaluate with a closed form.

5.2 Moments, Mass, and Moments of Inertia

Double integrals are useful in computing the centroids of certain lamina (surfaces). Given some lamina that is defined by the region R with an area density function given by $\rho(x, y)$, the mass of the lamina is given by the expression

$$M = \iint_R \rho(x, y) dA$$

The computation of this double integral, of course, almost always requires a conversion to an iterated integral system.

In physics, the moment of an object refers to the turning effect of a force about its balance point, its centroid. The moment of some point P is the product of its distance to its balance point along some axis and its density. So a moment of a lamina with respect to the x axis is expressed as

$$M_x = \iint y\rho(x, y) dA$$

and similarly, for the y axis,

$$M_y = \iint x\rho(x, y) dA$$

These quantities use the *signed* distance with respect to each axis. It is worth mentioning that symmetry can be used extensively in these calculations, as regions that are symmetric about an axis may present opportunities for simplification.

The balance points with respect to each axis of the lamina are the moments divided by the mass.

$$\overline{x} = \frac{M_y}{M}$$

$$\overline{y} = \frac{M_x}{M}$$

The coordinates of the centroid of the lamina, therefore, are

$$\left(\frac{M_y}{M}, \frac{M_x}{M}\right)$$

The moment of inertia of an object refers to its resistance against being rotated (the analogue of mass in a rotational context). The moment of inertia of the lamina with respect to the x axis is

$$I_x = \iint_R y^2 \rho(x, y) dA$$

and similarly,

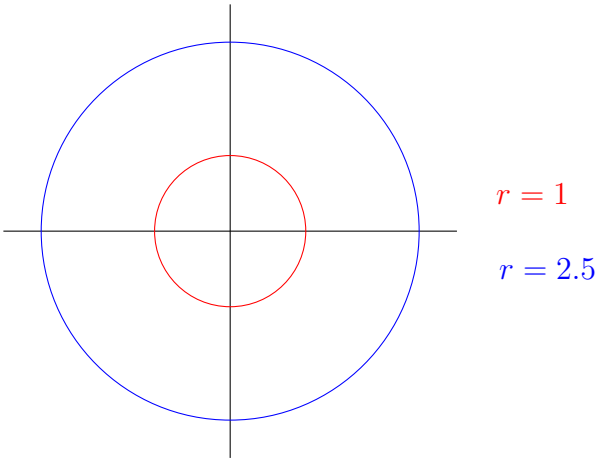
$$I_y = \iint_R x^2 \rho(x, y) dA$$

With respect to the origin,

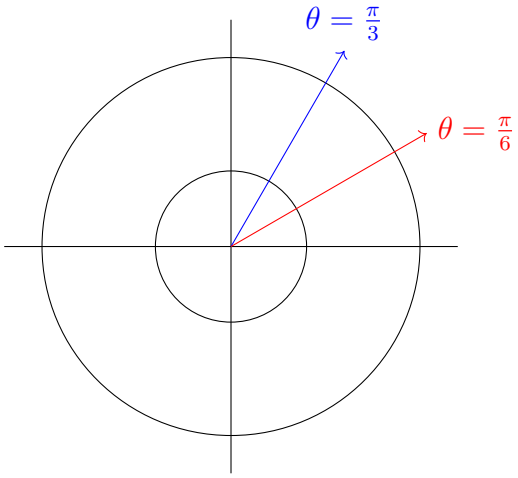
$$I_O = \iint_R (x^2 + y^2) \rho(x, y) dA = I_x + I_y$$

5.3 Polar Integration

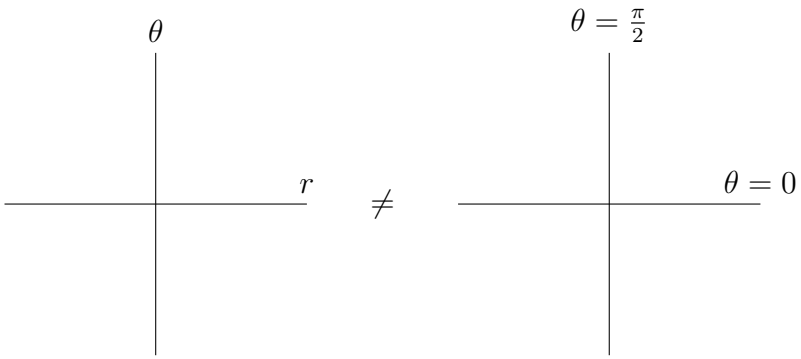
On a 2-D surface, a polar region is defined by angles, which are represented as rays that come from the "pole", or the origin of the polar graph, and radii, which are usually found as functions of angles. For example, the most elementary polar functions are circles, where the radius is defined as a constant:



Defining angle bounds on a polar region is like cutting a wedge-shaped slice out of a cylindrical cake.



To perform double integration given polar bounds, it seems tempting to simply replace x and y with r and θ ($dxdy \to drd\theta$). However, doing so would yield a rectangular region, and not the polar slice that we need.



You'll notice that, for polar regions, the area of each "block" of the partition gets larger, so we need some extra factor to account for this dilation. As it turns out, this factor is just r. So, a double integral of some polar region A will have the differential piece

$$dA = rdrd\theta = rd\theta dr$$

depending on the chosen order of integration. This gives us a change of variable formula:

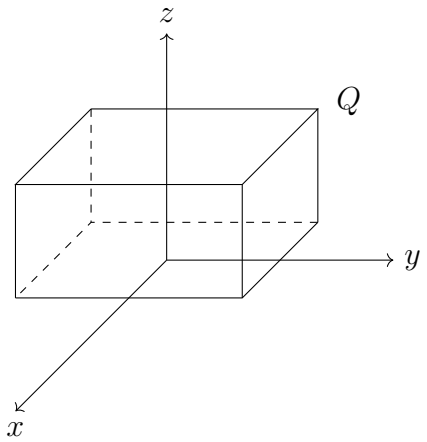
$$\iint_R f(x,y)dA = \int_{\theta_1}^{\theta_2} \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta,\ r\sin\theta)r\ dr\ d\theta$$

Or, in a similar vein,

$$\iint_R f(x,y)dA = \int_{r_1}^{r_2} \int_{h_1(r)}^{h_2(r)} f(r\cos\theta, r\sin\theta)r\ d\theta\ dr$$

6 Triple Integration

Given some 3-D region Q, it is possible to define triple integration in a manner that is similar to double integration.



We can take tiny partitions of rectangular prisms and add up their respective values, which leads us to the notion of a Riemann sum of a partition:

$$\sum_{i=1}^{n} f(u_i, v_i, w_i)\Delta V_i$$

for the ordered tuple (u_i, v_i, w_i) representing a point in Q_i. Taking a limit yields the formal definition of a triple integral over the region Q:

$$\iiint_Q f(x,y,z)dV = \lim_{||P||\to 0} \sum_i f(u_i, v_i, w_i)\Delta V_i$$

Furthermore, we can extend Fubini's theorem:

$$\iiint_Q f(x,y,z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\ dz\ dy\ dx$$

or something of a similar form. Now, how do we define the bounds on the iterated integrals? It's often helpful to think of the 3-D region as the end result of a definite integral inside of a double integral:

$$\iiint_Q f(x,y,z)dV = \iint_R \left[\int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)dz \right] dA$$

In this case, R would encompass the entire "shadow" of our figure onto the xy plane, or the largest area a cross section ("slice") along the z direction would attain.

6.1 Physics Applications

6.1.1 Moments and Mass

The mass of a 3-D object with density defined as $\rho(x,y,z)$ is equal to

$$M = \iiint_Q \rho(x,y,z)dV$$

In addition, we define the moments of the solid with respect to specific coordinate planes, rather than just the axes as we did in double integration.

$$M_{xy} = \iiint_Q z\rho(x,y,z)dV$$

$$M_{xz} = \iiint_Q y\rho(x,y,z)dV$$

$$M_{yz} = \iiint_Q x\rho(x,y,z)dV$$

Doing so allows us to find the center of mass of our object, which would be the coordinate

$$\left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M} \right)$$

6.1.2 Moments of Inertia

Moments of inertia are defined with respect to specific coordinate axes.

$$I_z = \iiint_Q (x^2 + y^2)\rho(x, y, z)dV$$

$$I_y = \iiint_Q (x^2 + z^2)\rho(x, y, z)dV$$

$$I_x = \iiint_Q (y^2 + z^2)\rho(x, y, z)dV$$

6.1.3 Gyration

The radius of gyration is a constant d such that

$$I = d^2 M$$

More concisely,

$$d = \sqrt{\frac{I}{M}}$$

6.2 Cylindrical Coordinates

Recall that a cylindrical coordinate system simply just adds a z coordinate to a 2-D polar system. It is described by the ordered tuple

$$(r, \theta, z)$$

where r is the radius on the 2-D plane, θ is the angle from the horizontal, and z is the elevation.

$$(x, y, z) \Rightarrow (r\cos\theta,\ r\sin\theta,\ z)$$

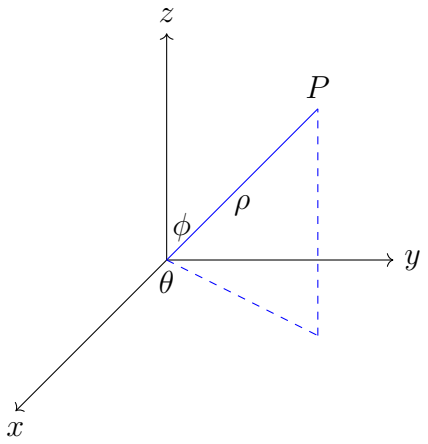
Performing a triple integral in a region defind by cylindrical coordinates is fairly simple; all we need to do is add a definite integral to our 2-D polar integral.

$$\iint_R \left[\int_{z_1}^{z_2} f(r, \theta, z)dz\right] \overset{(rdrd\theta)}{dA}$$

Setting up integrals in this kind of system is particularly helpful for cylinder-adjacent regions.

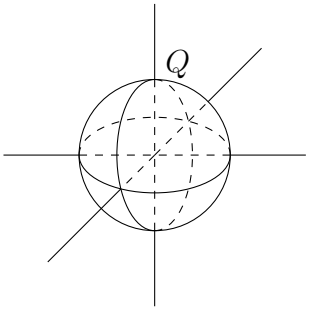
6.3 Spherical Coordinates

Spherical coordinates operate with a single radius (ρ), an angle of "sweep" (θ), and an angle of elevation (ϕ) from the vertical.



$$(x, y, z) \Rightarrow (\rho\cos\theta\sin\phi,\ \ \rho\sin\theta\sin\phi,\ \ \rho\cos\phi)$$

Analogous to the way we needed a factor of r in our polar double integral, we need a factor of $\rho^2\sin\phi$ to perform a coordinate change from cartesian to spherical coordinates. So, for a shape (sphere) like



$$\iiint_Q f(\rho,\theta,\phi)dV = \int_{\rho_1}^{\rho_2}\int_{g_1(\rho)}^{g_2(\rho)}\int_{h_1(\rho,\theta)}^{h_2(\rho,\theta)} f(\rho,\theta,\phi)\rho^2\sin\phi\ d\phi\ d\theta\ d\rho$$

Spherical coordinates are useful for sphere and cone-adjacent shapes.

7 Integral Manipulations

7.1 Surface Area

Recall that some curve $\mathcal{C}$ is represented as a parameterized vector with a single parameter,

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$$

Analogously, a surface $\mathcal{S}$ is represented as a parameterized vector with two parameters,

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$$

Note that surfaces defined as equations, like quadric surfaces, have simply undergone the process of eliminating the parameter, like we can do with parametric equation in 2-D; every surface can be described as a parameterized vector, but not every parameterized vector surface can be expressed as a single equation. We can introduce the concept of taking partial derivatives of vectors; as the vector for a surface is now composed of multivariable functions, we can take partial derivatives, which are still component-wise operations.

$$\mathbf{r}_u = \langle x_u(u, v), y_u(u, v), z_u(u, v) \rangle$$

$$\mathbf{r}_v = \langle x_v(u, v), y_v(u, v), z_v(u, v) \rangle$$

The surface area of a curve is given by the expression

$$\iint_{R\ (u,v)} |\mathbf{r}_u \times \mathbf{r}_v| \overset{(dudv)}{dA}$$

Intuitively, this can be thought of as summing up tiny pieces of parallelogram formed by the partials of the parameterized vector surface at a given point.

7.2 Change of Coordinates

There exist situations in which it is more convenient to integrate in a different coordinate system. For a given expression, we can define a transformation T such that

$$u = h(x, y)$$
$$v = k(x, y)$$

where $u,\ v$ are the bases of our new coordinate system. Recall from single-variable calculus the idea of a u-sub, often employed to simplify complicated integrals.

$$\int f(x)dx \Rightarrow \int f(g(u))g'(u)du$$

The scaling factor of some $g'(u)$ was justified by the idea of using differentials to "dilate" the area we were integrating with respect to; this scaling factor is the motivation behind seeking a "scaling factor" for multivariable changes of coordinates. For the inverse transformation of T (T^{-1})

$$x = f(u, v)$$

$$y = g(u, v)$$

the Jacobian is defined as the determinant of the matrix containing the first partials of f and g:

$$J_T = \frac{\partial(f, g)}{\partial(u, v)} = \begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix} = f_u g_v - f_v g_u$$

The Jacobian serves as the specific scaling factor for our double integral.

$$\iint_R F(x, y) \overset{(dxdy)}{dA} \Rightarrow \iint_R F(f(u, v), g(u, v))J_T \overset{(dudv)}{dA}$$

Note that, to perform an integral with a transformation in coordinates, T must be a 1-to-1 operation. Bounds should be kept in mind when performing these transformations; often, it is helpful to sketch out and graph specific curves and points as the transform from the xy plane to the uv plane.

We may also perform transformations in three dimensions. For a transformation

$$x = f(u, v, w)$$

$$y = g(u, v, w)$$

$$z = h(u, v, w)$$

the corresponding Jacobian is the following expression:

$$J_T = \begin{vmatrix} f_u & f_v & f_w \\ g_u & g_v & g_w \\ h_u & h_v & h_w \end{vmatrix}$$

which we may insert in front of our differential piece in computing the transformed integral.

8 Vector Calculus

8.1 Vector Fields

A vector field is defined as a relation

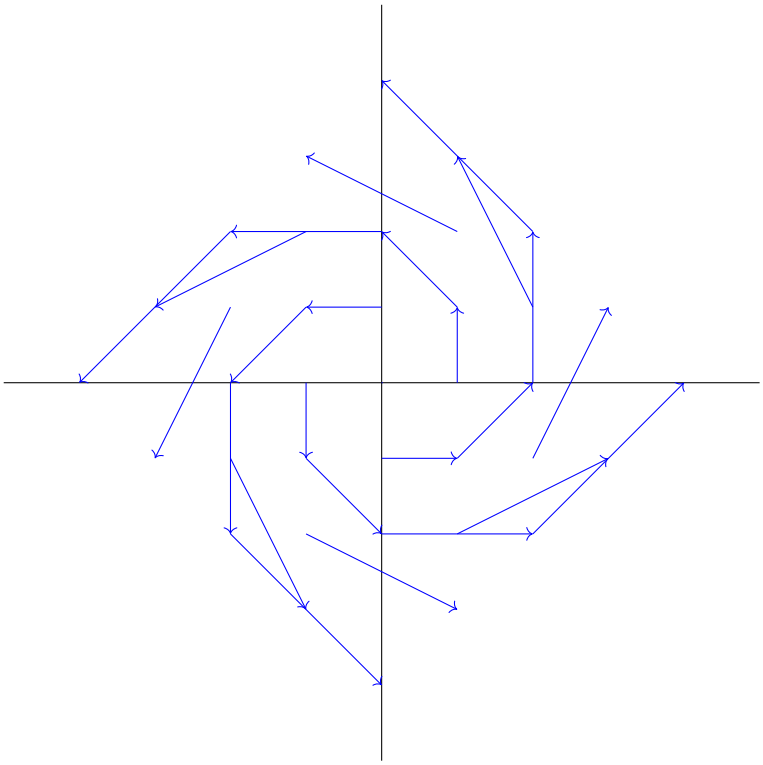
$$\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$$

$$\mathbf{F} = \langle M(x, y, z), N(x, y, z), P(x, y, z) \rangle$$

Essentially, it "assigns" every point in a space a specific vector, with an initial point at the specified location. For example, the field

$$\mathbf{F} = \langle -y, x \rangle$$

may look something like



and so on and so forth. There are a couple of important operations that we can perform with vector fields. Recall the definition of del:

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

For a vector function $\mathbf{F}$, we may dot or cross with ∇ (though the cross product is only defined in 3 and 7 dimensions). The operation

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F}$$

is defined as the *divergence* of $\mathbf{F}$: intuitively, it measures the "explosiveness" of the vector field about some central point, usually the origin.

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F}$$

is defined as the *curl* of $\mathbf{F}$, indicating how much the vector field "spins" about some location.

A *conservative* vector field is a field for which there exists some function f such that $\nabla f = \mathbf{F}$. For example,

$$\mathbf{F} = \langle y, x \rangle$$

is a conservative vector field because it is the gradient of the function

$$f = xy + C$$

for any arbitrary constant.

8.2 Path Integrals

Path integrals, also known as line and contour integrals, is the concept of a tradition integral instead performed on some arbitrary curve, instead of the x axis. For an $f(x, y)$ defined on a curve $\mathcal{C}$, we can divide the curve into small arcs of length Δs. By choosing a point within each subarc and multiplying by the value of the function, we come to the sum

$$\sum f(u_i, v_i) \Delta s_i$$

which, after taking a limit as $||\Delta s_i||$ goes to 0, yields the integral of f over the curve $\mathcal{C}$.

$$\int_{\mathcal{C}} f(x, y) ds = \lim_{||\Delta|| \to 0} \sum_i f(u_i, v_i) \Delta s_i$$

From single variable calculus, we know that for a smoothly parameterized curve,

$$s = \int_a^b \sqrt{[g'(t)]^2 + [h'(t)]^2} dt$$

Thus, we can see that

$$ds = \sqrt{[g'(t)]^2 + [h'(t)]^2}dt$$

and so

$$\int_{\mathcal{C}} f(x,y)ds = \int_a^b f(g(t),h(t))\sqrt{[g'(t)]^2 + [h'(t)]^2}dt$$

Notice that this can be written in vector notation as well:

$$\int_{\mathcal{C}} f(x,y)ds = \int_a^b f|_{\mathbf{r}(t)} \cdot |r'(t)|dt$$

We can add path integrals, as well: for a piecewise smooth curve $\mathcal{C}$ consisting of a finite number of smooth curves $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$, then the path integral over the entire curve is the sum of the path integrals within each of its parts.

However, this method of taking path integrals of scalar functions is extremely limiting, and needing to take the square root in the differential piece can introduce computational challenges. Instead, there exists a more general definition of a path integral for a vector field function $\mathbf{F}$.

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}|_{\mathbf{r}} \cdot \mathbf{r}'(t)dt$$

Notice that this can be turned into something of the form

$$\int_a^b M(x,y)dx + N(x,y)dy$$

where dx and dy can be transformed into some dt based on the parameterization of the curve. This is how path integrals often appear in practice. Notice that this equation is exactly the one used to calculate work for a non-constant vector force; path integrals are extremely useful in mechanics.

8.2.1 Fundamental Theorem of Path Integrals (FToPI)

For a path $\mathcal{C} : \mathbf{r}(t)$ and a conservative vector field $\mathbf{F}$ ($\mathbf{F} = \nabla f$) on an open domain $\mathcal{D}$ such that $\mathcal{C} \subseteq \mathcal{D}$, then

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = f|_{\mathbf{r}(a)}^{\mathbf{r}(b)}$$

The following statements are equivalent:

1. $\mathbf{F}$ is a conservative vector field
2. $\exists f \ s.t. \ \mathbf{F} = \nabla f$
3. $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is path-independent
4. $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 0$
5. $N_x = M_y$

However, the last statement is only applicable for two dimensions.

8.2.2 Center of Mass

Path integrals can be used to find the moments and center of mass for some linear mass with a density expressed by $f(x, y)$. The moments with respect to the x and y axes, respectively, are defined as

$$M_x = \int_{\mathcal{C}} yf(x, y)ds$$

$$M_y = \int_{\mathcal{C}} xf(x, y)ds$$

Analogous to earlier discussions of center of mass,

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right)$$

8.3 Surface Integrals

A surface integral can be thought of as a 2-dimensional analogue of a path integral. Intuitively, given some surface $\mathcal{S}$, we split it up into tiny regions of $d\mathcal{S}$. Multiplying by the function evaluated at some point within the tiny region, then, will yield a summation of components, which can be transformed into a double integral through a limit.

$$\iint_{\mathcal{S}} f(x, y, z)d\mathcal{S} = \lim_{||\Delta|| \to 0} \sum_i f(x_i, y_i, z_i)\Delta T_i$$

where ΔT_i denotes small parts of the tangent plane to a surface. We know that the surface area of $\mathcal{S}$ cadn be evaluated through a double integral:

$$\text{Area} = \iint_R |\mathbf{r}_u \times \mathbf{r}_v| dA$$

for $\mathcal{S} : \mathbf{r}(u,v) = \langle g(u,v), h(u,v), k(u,v)\rangle$. Thus,

$$d\mathcal{S} = |\mathbf{r}_u \times \mathbf{r}_v| dA$$

and so

$$\iint_{\mathcal{S}} f(x,y,z) d\mathcal{S} = \iint_R f(x,y,z) \cdot |\mathbf{r}_u \times \mathbf{r}_v| dA$$

This is the version of a surface integral for a scalar surface. However, taking a cross product and then its magnitude can be annoying, at times, and this form of a path integral can be limiting for several important use cases.

We introduce the concept of a vector surface integral: for a vector field $\mathbf{F}$ and a surface $\mathcal{S}$, the surface integral of $\mathbf{F}$ over $\mathbf{S}$ is

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n} d\mathcal{S} = \iint_R \mathbf{F}|_{\mathbf{r}(u,v)} \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA$$

This quantity is also known as the *flux* of $\mathbf{F}$ over $\mathcal{S}$, denoted

$$\Phi = \iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n} d\mathcal{S}$$

It should be noted that, for closed surfaces (that is, surfaces that contain some volume Q), the normal vector $\mathbf{n}$ must point *out* of the shape.

9 Fundamental Theorems

9.1 Green's Theorem

For a positively oriented, finitely piecewise smooth, simple, closed, planar curve $\mathcal{C}$, with $R = \mathcal{C} \cup \text{int}(\mathcal{C})$ and an M and N with continuous first partials throughout some domain $\mathcal{D}$ such that $R \subseteq \mathcal{D}$, then

$$\oint_{\mathcal{C}} M dx + N dy = \iint_R N_x - M_y dA$$

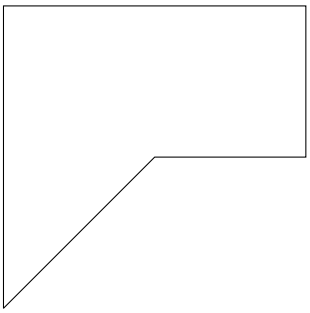
An aside about notation: the symbol $\oint$ indicates a path integral along some closed curve. In general, a ring on an integral symbol indicates that the region it is operating is closed.

A *positive orientation* indicates that we are traveling in a direction such that, if we envision ourselves to be walking around the closed loop, our left hand would be pointing towards the interior of the region.

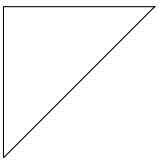
A corollary to Green's theorem states that the area under some parametrized graph can be expressed as

$$A = \oint_C x dy = -\oint_C y dx = \frac{1}{2}\oint x dy - y dx$$

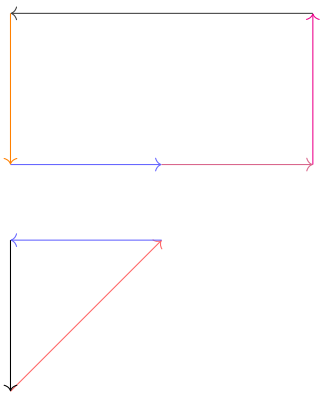
Green's theorem also works for multiple, connected regions. To see why, first consider a complex region that is neither type I or II:



Green's theorem says that the area of this shape should be equal to the path integral along the boundary. We can, then, split up our region into smaller, simpler regions of either type I or type II:



The path integral of each separate region should add up to the combined areas of the regions. We recall the requirement of a positive orientation in applying the theorem; we must walk in a direction such that our left arm "points" towards the center of the region.



Note how the two blue boundaries travel in "opposite" directions; the evaluation of one will be the negation of the other. Therefore, the theorem works for any combination of type I and II regions.

Note that the integrand for the double integral, $N_x - M_y$, looks familiar: it is the z component of curl $\mathbf{F}$, or curl $\mathbf{F} \cdot \mathbf{k}$. Thus, the vector form of Green's theorem looks like

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k} dA$$

9.2 Divergence Theorem

The Divergence theorem, also known as Gauss' theorem or Ostrogradsky's theorem in different contexts, states that for a closed surface $\mathcal{S}$ that encloses some volume Q,

$$\oiint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n} d\mathcal{S} = \iiint_Q \nabla \cdot \mathbf{F} dV$$

Analogous to Green's theorem but in the context of surfaces, this theorem allows us to choose which method to use when computing either a surface or a volume integral, which may make things computationally easier. Notice that we may derive an interpretation of divergence from this theorem. $\nabla \cdot \mathbf{F}$ is effectively the "normalization" of flux surrounding some point P by the volume it takes up. For a circle of radius ε around some point P,

$$\nabla \cdot \mathbf{F} \approx \frac{1}{V_\varepsilon} \iint_{\mathcal{S}_\varepsilon} \mathbf{F} \cdot \mathbf{n} d\mathcal{S}$$

9.3 Stokes' Theorem

Recall the vector form of Green's theorem:

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k} dA$$

Stokes' theorem can be thought of as the generalization of Green's theorem to all curves, not just planar ones.

For a surface $\mathcal{S}$ that is finitely piecewise smooth, oriented, and bounded by a curve $\mathcal{C}$ that is also finitely piecewise smooth, simple, closed, and positive oriented, and $\mathbf{F}$ with continuous first partials on an open region containing $\mathcal{S}$,

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{S}} \nabla \times \mathbf{F} \cdot \mathbf{n} d\mathcal{S}$$

Note the consequences of this statement: it doesn't matter what the surface we are integrating over is, as long as its outermost boundary remains the same. From this law, we can glean the idea that curl represents the tendency of a vector field to rotate about some normal vector $\mathbf{n}$. For a circle of radius ε around some point P,

$$\nabla \times \mathbf{F} \approx \frac{1}{\pi \varepsilon^2} \oint_{\mathcal{C}_\varepsilon} \mathbf{F} d\mathbf{r}$$

We revise the last statement in our TFAE for path integrals; the true, general statement instead involves curl:

$$\text{curl } \mathbf{F} = \mathbf{0}$$