Analysis 1

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Contents

1 Limits

1.1 Definition of a Limit

The limit of a function as it goes to a certain point a is a value L as long as the values defined in the function around the value a get closer and closer towards L. By this definition, a does not have to be within the domain of f, as long as values around it are.

Formally: let $f(x)$ be a function defined on an interval that contains $x = a$. We may say that

$$\lim_{x \to a} f(x) = L$$

if, for every number $\varepsilon > 0$ there is a $\delta > 0$, such that $|f(x) - L| < \varepsilon$ when $0 < |x - a| < \delta$.

The limit of a function at a point c does not exist if $\lim_{x \to c^-} \neq \lim_{x \to c^+}$ — that is, if both sides of the limit do not agree, then the limit does not exist. Note that a limit can still exist even if the point $f(c)$ does not; functions can agree except at a point.

1.2 Properties of Limits

For functions $f(x), g(x)$, a constant c and some real numbers a, n:

$$\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x) \tag{1}$$

$$\lim_{x \to a} f(x) \pm g(x) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) \tag{2}$$

$$\lim_{x \to a} f(x)g(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) \tag{3}$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \tag{4}$$

$$\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n \tag{5}$$

$$\lim_{x \to a} c = c \tag{6}$$

$$\lim_{x \to a} x = a \tag{7}$$

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)) \tag{8}$$

$$\tag{9}$$

1.3 Squeeze Theorem

Define three functions such that $f(x) \leq g(x) \leq h(x)$. Then, if

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L,$$

then

$$\lim_{x\to a} g(x) = L.$$

The squeeze theorem is particularly useful when it comes to sine/cosine functions, that already come with given bounds. For example, to find $\lim_{x\to\infty} \frac{\sin x}{x}$, first note that $-1 \leq \sin x \leq 1$.

$$\Rightarrow -\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

$$\Rightarrow -\lim_{x\to\infty} \frac{1}{x} \leq \lim_{x\to\infty} \frac{\sin x}{x} \leq \lim_{x\to\infty} \frac{1}{x}$$

$$\Rightarrow 0 \leq \lim_{x\to\infty} \frac{\sin x}{x} \leq 0$$

$$\Rightarrow \lim_{x\to\infty} \frac{\sin x}{x} = 0$$

1.4 Evaluation

Plug it in Sometimes, limits are simple enough that merely plugging in for the variable lets reveals the limit.

Factoring Common within limits with rational functions, we may factor the numerator and the denominator and attempt to cancel terms out, in order to manipulate the function to provide a deternminate answer when the variable is plugged in. With limits, functions may be equal except at a point; this is perfectly legal.

Multiplying by 1 We may also multiply by some version of "1", the multiplicative identity. A case where this concept is incredibly useful is when we require the use of conjugate pairs; especially if we had a square root or a complex number that is difficult, or impossible, even, to work with.

Divide by the degree of the leading term This holds only for limits to infinity. When we have some polynomial expression, we may divide each term in the expression by the degree of the leading term; the leading coefficients will remain, which will be the result of the limit.

1.5 Epsilon-Delta Proofs

Proving a limit with the epsilon-delta definition is usually done in two "steps" — a discovery phase and a formal production of the proof.
Let $f(x) = x + 7$. We will try to prove the statement: $\lim_{x\to 2} f(x) = 9$.

1.5.1 Discovery

To find this limit, we may obviously substitute x within the equation with 2 — however, we have not proved this result rigorously. To utilise the formal definition of a limit, suppose that for every epsilon in which $|f(x) - 9| < \varepsilon$, there is at least one delta such that $0 < |x-2| < \delta$. Here, our goal is to prove that such a delta exists for all epsilon close to our limit of 9. To achieve the inequalities we desire, we may do some algebraic manipulation:

$$|f(x) - 9| < \varepsilon$$

$$|x + 7 - 9| < \varepsilon$$
$$|x - 2| < \varepsilon$$

Since we have ended up with the same exact expression with our inequality for delta, what we see is that we may choose $\delta = \varepsilon$.

1.5.2 Formal Proof

To prove: $\lim_{x\to 2} f(x) = 9$ Assume ε is a positive real number and choose $\varepsilon = \delta$
We may set up our inequalities, as per the definition of a limit:

$$0 < |x - 2| < \delta = \varepsilon$$

$$0 < |(x + 7) - 9| < \varepsilon$$
$$|f(x) - 9| < \varepsilon$$

1.5.3 Quadratic Limits

Quadratic limits are more challenging than the standard linear limit proof because there is an extra term that cannot be dealt with, provided that the expression is not factorable into a perfect square. Let $f(x) = x^2$. To prove that $\lim_{x\to 2} f(x) = 4$, we may begin with a standard setup:

$$|f(x) - 4| < \varepsilon$$

$$|x^2 - 4| < \varepsilon$$

$$|x + 2||x - 2| < \varepsilon$$

Now, it is clear that we are close to our ultimate goal of reaching the inequality $0 < x - 2 < \delta$. However, we still have the term $|x + 2|$ to take into consideration. To deal with this, we can attempt to show that $|x + 2| < N$ for some real number N. Then, we will only have to show:

$$|x - 2| < \frac{\varepsilon}{N}.$$

In order to achieve this, we recall the fact that we only care about values close to $x = 2$, say, with a difference of 1.

$$-1 < x - 2 < 1$$

$$3 < x + 2 < 5$$

Thus, we may set $N = 5$ because we know that $|x + 2| < N$ for all such values that we care about, close to $x = 2$. To recap, our assumptions so far are:

$$|x - 2| < 1$$

$$|x - 2| < \frac{\varepsilon}{5}$$

In order to construct our proof, we may just choose $\delta = \min(1, \frac{\varepsilon}{5})$.

1.5.4 Finding a δ Given ε

These types of problems require you to find the maximum difference in the x value such that the resulting output is less than a given ε. Given $\varepsilon = 0.01$, find the minimum δ for $\lim_{x\to 2} x^2 - 3 = 1$.

$$|x^2 - 3| - 1 < 0.01, 0 < |x - 2| < \delta$$

$$3.99 < x^2 < 4.01, 0 < |x - 2| < \delta$$

$$\sqrt{3.99} < x - 2 < \sqrt{4.01} - 2, \delta = \min(\sqrt{3.99} - 2, \sqrt{4.01} - 2)$$

1.6 Infinite Limits

1.6.1 At Infinity

The limit of a function as it goes towards infinity can be classified as the end behavior of the function. As stated before, we may divide by the leading term of the denominator to find the limit of the function as it goes towards positive or negative infinity, if the limit exists. If $\lim_{x\to\pm\infty} = L$, then $y = L$ is a horizontal asymptote of $f(x)$. Else, the function grows without bound. For rational functions, we may examine the leading exponents: $\frac{ax^m}{bx^n}$

- If $m > n$, there is no horizontal asymptote and the function grows without bound.
- If $m = n$, there is a horizontal asymptote at $y = \frac{a}{b}$.
- If $m < n$, there is a horizontal asymptote at $y = 0$.

1.6.2 Approaching Infinity

If $\lim_{x\to c} \to \infty$, then the function extends without bound at that point.

1.7 L'Hopital's rule

When dealing with limits, there are seven indeterminate forms:

$$1^\infty,\ 0^0,\ \infty^0,\ \infty - \infty,\ \tfrac{0}{0},\ \tfrac{\infty}{\infty},\ \infty \cdot 0$$

In addition to these, it is helpful to state when to **not** use L'Hopital's rule:

$$\infty + \infty = \infty,\ -\infty - \infty = -\infty,\ 0^\infty = 0,\ 0^{-\infty} = \infty,\ \infty \cdot \infty = \infty$$

L'Hopital's rule states, for the following cases:

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{0}{0}$$

or

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$$

where $c \in \mathbb{R}, \infty, -\infty$:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}$$

It is important to note that this only works if the limit is of an indeterminate form. We may execute several manipulations to reach a form $\frac{f(x)}{g(x)}$.

Example: Exponents $k = \lim_{x \to 0^+} x^x$

$$\begin{aligned} \ln k &= \ln \lim_{x \to 0^+} x^x \\ &= \lim_{x \to 0^+} x \ln x \end{aligned}$$

$$\lim_{x \to 0^+} x \ln x \to 0 \cdot -\infty$$

$$\lim_{x \to 0^+} \frac{\ln x}{x^{-1}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-x^{-2}}$$

$$= \lim_{x \to 0^+} -x = 0$$

$$\ln k = 0, k = 1$$

2 Continuity

2.1 Definition of Continuity

A function $f(c)$ is defined to be continuous in an interval (a, b) if, for $\forall c \in (a, b)$, it upholds the condition:

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$$

Note that the existence of $\lim_{x \to c} f(x)$ implies that the limit exists from both sides.

2.2 Intermediate Value Theorem

For a function $f(x)$ that is continuous on the interval $[a, b]$ as well as an $N \in [f(a), f(b)]$: There exists such a value c such that $c \in (a, b)$ and $f(c) = N$. This theorem can be used to prove that specific values of functions exist, provided that the function is continuous on the defined interval.

2.3 Consequences

If a function is differentiable at some point, then it is also continuous. However, it does not commute: continuity does not necessarily imply differentiability. Continuity implies integrability, but integrability does not imply continuity.

3 Derivatives

3.1 Definition of a Derivative

The instantaneous rate of change for a function at a point is defined as the limit as a secant line of a function between two points becomes a tangent line. In other words, as a point at a moves closer to b, the slope of the line between the points converges upon the true tangent line slope at the point b. We may express this as:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

This adheres to the definition of a slope — $\frac{rise}{run}$. We may generalize this as an equation for the tangent line of all points with

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

3.2 Properties

For functions $f(x), g(x)$ and a constant c:

$$\frac{d}{dx}[cf(x)] = cf'(x) \qquad \text{(Constant Multiplication)}$$

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) + g'(x) \qquad \text{(Addition)}$$

$$\frac{d}{dx}[x^n] = nx^{n-1} \qquad \text{(Power Rule)}$$

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x) \qquad \text{(Product Rule)}$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \qquad \text{(Quotient Rule)}$$

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x) \qquad \text{(Chain Rule)}$$

3.3 Proofs

3.3.1 Power Rule

$$\begin{aligned} y &= x^n \\ \ln y &= n \ln x \\ \frac{y'}{y} &= \frac{n}{x} \\ y' &= \frac{n \cdot y}{x} \\ y' &= x^n \cdot \frac{n}{x} \\ y' &= nx^{n-1} \end{aligned}$$

3.3.2 Product Rule

$$\begin{aligned} y &= f(x)g(x) \\ \ln y &= \ln f(x) + \ln g(x) \\ \frac{y'}{y} &= \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)} \\ y' &= (f(x)g(x)) \left(\frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)} \right) \\ y' &= g(x)f'(x) + f(x)g'(x) \end{aligned}$$

3.3.3 Quotient Rule

$$
\begin{aligned}
y &= \frac{f(x)}{g(x)} \\
\ln y &= \ln f(x) - \ln g(x) \\
\frac{y'}{y} &= \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)} \\
y' &= \left(\frac{f(x)}{g(x)}\right)\left(\frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}\right) \\
y' &= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{(g(x)^2)} \\
y' &= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}
\end{aligned}
$$

3.4 Implicit Differentiation

Instead of isolating for a single variable in an equation, implicit differentiation just uses the chain rule to expand when a derivative is taken on both sides. For example:

$$x^2 + y^2 = 1$$

If we decide to isolate for y, we end up with a difficult expression. Instead, we can just differentiate with respect to x on both sides:

$$2x + 2y \cdot y' = 0$$

To get y' in terms of x, we may perform simple algebraic manipulations and substitutions.

3.4.1 Logarithmic Differentiation

Logarithmic differentiation is a form of implicit differentiation, used mostly when there is an expression in an exponent. With the equation:

$$y = (x^2 + 2x + 1)^x$$

we may apply a natural logarithm on both sides to bring down the exponent.

$$\ln y = x \ln(x^2 + 2x + 1)$$

To find $\frac{dy}{dx}$, we may just apply implicit differentiation:

$$\frac{1}{y} \cdot y' = \ln(x^2 + 2x + 1) + x \cdot \frac{2x + 2}{x^2 + 2x + 1}$$

3.5 Theorems

3.5.1 Extreme Value Theorem

For a function $f(x)$ continuous on a closed interval $[a, b]$, f must have a miniumum and maximum on the interval, where c, d represent the location of the lowest and highest point, respectively.

$$f(c) \leq f(x) \leq f(d), \forall x \in [a, b]$$

3.5.2 Mean Value Theorem

If a function $f(x)$ is continuous and differentiable on the closed interval $[a, b]$, then there is at least one point $c \in (a, b)$ where

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

3.5.3 Rolle's Theorem

A special case of the Mean Value Theorem: for a function $f(x)$ continuous and differentiable on a closed interval $[a, b]$ where $f(a) = f(b) = 0$, there is some point $c \in (a, b)$ such that

$$f'(c) = 0$$

3.5.4 Fermat's Theorem

If $f(x)$ has a relative extrema at $x = c$, then c is a critical point, otherwise defined as when $f'(x) = 0$ or doesn't exist.

3.6 Relative and Absolute Extrema

3.6.1 Minima and Maxima

- A function $f(x)$ has an absolute maximum at $x = c$ if $f(x) \leq f(c)$ within the domain of interest of f.

- Conversely, a function $f(x)$ has an absolute minimum at $x = c$ if $f(x) \geq f(c)$ within the domain of interest of f.

- $f(x)$ has a relative maximum at $x = c$ if $\forall x, f(x) \leq f(c)$ for some open interval (a, b) around c.

- Conversely, $f(x)$ has a relative minimum at $x = c$ if $\forall x, f(x) \geq f(c)$ for some open interval (a, b) around c.

3.6.2 Critical Points

For a function $f(x)$, critical points are located at any point c such that $f'(c) = 0$ or undefined. Note that c must exist inside the domain of the function. All relative extrema are located at critical points.

3.6.3 Not The First Derivative Test

Critical points may be found where $f'(x)$ is either undefined or is 0, and indicate where a function changes from increasing to decreasing, or vice versa. By testing values all throughout the function, one can conclude its sloping behavior at that point with the NTFDT: if the $f'(x) > 0$, then the function is increasing; vice versa, if $f'(x) < 0$, then the function is decreasing.

3.6.4 First Derivative Test

Based on the information from the NTFDT, extrema occur where the sloping behavior changes sign. If the sign changes from negative to positive, the point is a minimum. If the slope changes from positive to negative, then the point is a maximum. Note that points that are undefined within the function cannot be extremum.

3.6.5 Inflection Points

Inflection points occur when the concavity of a function changes; indicated otherwise when the second derivative of a function changes sign. Note that the sign of the second derivative must change for there to be an inflection point.

3.6.6 Not The Second Derivative Test

On any interval between inflection points, if $f''(x) > 0$, then f is concave up on that interval - like a cup. If $f''(x) < 0$, then f is concave down.

3.6.7 Second Derivative Test

- If $f'(c) = 0$ and $f''(c) > 0$, then there is a minimum at $x = c$.
- If $f'(c) = 0$ and $f''(c) < 0$, then there is a maximum at $x = c$.
- If $f'(c) = 0$ and $f''(c) = 0$ or does not exist, then the test is indeterminate.

3.6.8 Notes and Remarks

The distinction behind the Not The Derivative tests and the Derivative tests is that Not The Derivative tests find extrema and say something about the behavior of the function, while the Derivative tests classify those extrema as either minima or maxima.

3.7 Linearization and Approximation

3.7.1 Differentials

With a function $y = f(x)$, the differentials of the function are defined by the relationship

$$dy = f'(x)dx$$

We can use differentials to calculate error in approximating functions. For an independent variable, the differential is just equal to its change, or Δx.

3.7.2 Newton-Raphson Method

The Newton-Raphson method is a way to find the zeroes of a function by linearizing it based on the derivative at a point. It relies on multiple iterations of the same equation:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Failures:

- May get stuck at relative extrema
- Fails when derivative at any point is 0
- May diverge
- First guess must be near desired root
- May oscillate around a point

3.8 Applications

3.8.1 Optimization

If we define a function $f(x)$ that describes a specific "cost" of a system, then the most "optimal" set of parameters will occur at endpoints, minima or maxima, depending on the specific situation at hand. For example, if we wanted to maximize the enclosed area provided by 625 feet of fence, we may first set up a constraint equation and a function for optimization: $2x + 2y = 500$, and we want to maximize xy.

$$y = 250 - x, A = x(250 - x)$$

$$A = 250x - x^2$$

$$A' = 250 - 2x = 0, x = 125$$

$$y = 250 - x, y = 125$$

Our final, maximized area turns out to be $125^2 = 15625$. Checking the values surrounding the location such as the pair (124, 126) = 15624 reveal that the function is, indeed, maximized at (125, 125).

3.8.2 Related Rates

Related rates problems involve finding the rate at which one quantity changes in relation to another quantity whose rate of change is known. Many related rates problems begin with basic geometric formulas to construct relationships between variables. For example, if we know that $x^2+y^2 = 25$, with the initial conditions $x = 3, x' = 0.1, y = 4$, we may use implicit differentiation to solve for y':

$$\begin{aligned}
\frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}25 \\
2xx' + 2yy' &= 0 \\
y' &= \frac{-2xx'}{2y} \\
&= -\frac{xx'}{y} \\
&= -\frac{0.3}{4}
\end{aligned}$$

4 Integrals

4.1 Indefinite Integrals

An indefinite integral is defined to be the antiderivative of a function. That is, a function $F(x)$ is the antiderivative of $f(x)$ if $F'(x) = f(x)$. We notate this with:

$$\int f(x)dx = F(x) + c$$

with c being any constant. To perform an antiderivative, we can just "step back in time" and do the opposite of a derivative operation.

4.2 Properties of Indefinite Integrals

Some of the properties of derivatives extend to antiderivatives:

$$\int cf(x)dx = c\int f(x)dx$$

$$\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$$

4.3 Integration Techniques

4.3.1 U-Sub

When differentiating, we use the chain rule to deal with composite functions. Similarly, with integration, we may use a u-sub to solve for the expression. We recall that for a function $u(x)$, the differential can be expressed as $du = u'dx$. When we substitute some expression in the function with u, we may change the differential of the integral to align with the change in variable. For example, for a function $2x \cdot \cos x^2$:
We can substitute $u = x^2$, and therefore, $du = 2x \cdot dx$. Then, the integral of $\int 2x \cdot \cos x^2 dx$ may be expressed as:

$$\int 2x \cos u dx = \int \cos u du$$

$$= \sin u + c = \sin x^2 + c$$

4.3.2 Integration by Parts

To do integration by parts, we can use the following formula:

$$\int udv = uv - \int vdu$$

where we choose two functions that are being multiplied within the integral to change into the larger transform. We pick a u piece, which entails differentiating with respect to x to obtain a du differential piece, while with the v piece we integrate with respect to x to obtain just a v piece. For example:

$$\int x \cos x dx$$

We cannot simply use some u-sub to solve this problem. Instead, we must use parts:

$$u = x, du = dx, dv = \cos x dx, v = \sin x$$

This then becomes

$$x \sin x - \int \sin x dx = x \sin x + \cos x + c$$

4.3.3 Swingy-Swingy

A corollary to integration by parts is swingy swingy — where we obtain the original integral on the left side of our equation with different coefficients, after which we may "swing" said integral to the right hand, original side of the equation, allowing us to simply use algebraic manipulation to solve. Swingy swingy is commonly used with trigonometric functions as well as e^x, since they have cyclic derivatives. For example:

$$\int \cos x \cdot e^x dx$$

$$u = e^x, du = e^x dx, dv = \cos x dx, v = \sin x$$

$$e^x \sin x - \int \sin x \cdot e^x dx$$

$$u = e^x, du = e^x dx, dv = \sin x dx, v = -\cos x$$

$$e^x \sin x - (-e^x \cos x + \int e^x \cos x dx) = \int e^x \cos x dx$$

We ended up with the same integral in our equation as before.

$$e^x \sin x + e^x \cos x = 2 \int e^x \cos x dx$$

$$\int e^x \cos x = \frac{1}{2}(e^x \sin x + e^x \cos x)$$

4.3.4 OIGOIG — EFUSOT

OIGOIG stands for "Odd Is Good, Odd Is Great" with a corollary of "Even Sucks, Except For U-Subs Of Tangent". This is less of a technique and more of a piece of advice — when subbing for either $\cos x$ or $\sin x$, then we should choose the function with an odd power as the differential piece for our u-sub, which allows us to dissolve the expression cleanly with power-reducing identities. The inverse is true for u-subs of tangent, as stated in the name.

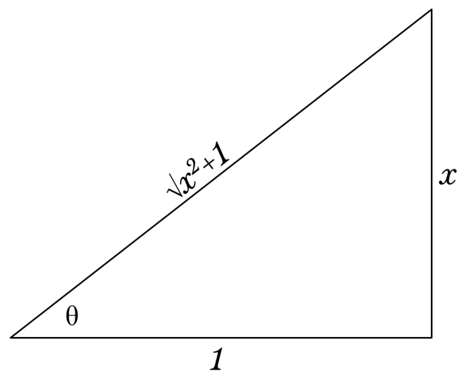
4.3.5 Partial Fractions

The method of partial fractions involves factoring the denominator and then setting up a "system" of equations that follow, leaving us with a nice additive equation of fractions, rather than multiples of them. There are some things to consider:

1. When decomposing with duplicate terms, we must create a separate denominator for each respective "power" of the term; for example, to decompose a fraction like $\frac{1}{(x+1)^2}$, we must set up an equation like $\frac{A}{x+1} + \frac{B}{(x+1)^2}$.
2. Sometimes, things don't factor nicely into simple x terms. In such a case, we must add coefficients for powers of x up to $n-1$, with n being the degree of the denominator. For example, with a fraction like $\frac{1}{(x+1)(x^2+1)}$, we need an expression like $\frac{A}{x+1} + \frac{Bx+C}{x^2+1}$.

4.3.6 Trigsub

Whenever the expression in question involves square roots of some variable squared plus a constant (e.g. $\sqrt{x^2+1}$), we should use trigsub. This simply involves setting up a triangle such that its sides correspond to the constant and the variable in question, and then using trigonometric functions to express the same function in terms of the angle θ. In addition, we must also find the differential piece $d\theta$ with respect to dx and substitute it in to obtain a complete and valid integral. For example, for the integral $\int \sqrt{x^2+1}dx$:



We have drawn the appropriate triangle — now, all we have to do is to substitute. We can begin by substituting $\sqrt{x^2+1}$ as $\sec\theta$. We also know that $x = \tan\theta$ — therefore, $dx = \sec^2\theta d\theta$. We can now substitute the entire integral:

$$\begin{aligned}\int \sqrt{x^2+1}dx &= \int \sec\theta \cdot \sec^2\theta d\theta \\ &= \frac{1}{2}(-\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|) \\ &= \frac{1}{2}(-\sec\arctan x \tan\arctan x + \ln|\sec\arctan x + \tan\arctan x|) \\ &= \frac{1}{2}(-x\sqrt{x^2+1} + \ln|\sqrt{x^2+1} + x|)\end{aligned}$$

Note: for simplicity's sake, I did not include the derivation of $\int \sec^3 x$ — it may be derived from integration by parts.

4.3.7 Crazy Tan

To use crazy-tan, we first sub $u = \tan\frac{\theta}{2}$. From this, we can derive the following differential piece and identities:

$$d\theta = \frac{2}{1+u^2}du$$

$$\cos\theta = \frac{1-u^2}{1+u^2}$$
$$\sin\theta = \frac{2u}{1+u^2}$$
$$\tan\theta = \frac{2u}{1-u^2}$$

...and the substitutions for other trig functions follow from there. If we plug in these identities into a particularly nasty integral involving trigonometric functions, then we obtain an equation made entirely of u pieces — which is to say, either a polynomial or a rational function, which is easier to work with.

4.4 Definition of a Definite Integral

A definite integral in terms of x with an even partition for a function of x can be defined as the signed area under the curve of the function, or, the summation of an infinite number of boxes constructed to fit the curve. In the expression, x_i^* simply means "the value of x at the increment i."

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Note: an integral is not necessarily the area under a curve. An integral is a sum of small pieces; if the entities being summed are boxes, then the integral provides the area.

While this is sufficient for regular partitions, when we are considering unequal partitions, we must add the condition that the maximum width of any piece $\to 0$. Let $||\Delta|| = \max(\Delta x_i)$:

$$\lim_{||\Delta||\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

4.5 Properties of Definite Integrals

$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^a f(x)dx = 0$$

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$$

4.6 Fundamental Theorem of Calculus

4.6.1 Part 1

For a function $f(x)$ continuous on the closed interval $[a, b]$ and $F(x) = \int f(x)dx$, then:

$$\int_a^b f(x)dx = F(b) - F(a)$$

Note: when we use a u-sub to solve definite integrals, we must remember to either change endpoints to match the u-differential, or indicate clearly that the endpoints are for a differential other than du.

4.6.2 Part 2

For a function $f(x)$ continuous on an interval I and with $a \in I$ if $F(x)$ is defined by:

$$F(x) = \int_a^x f(t)dt$$

then

$$F'(x) = f(x)$$

This statement is crucial because it relates derivatives and definite integrals. As for the more generalized version of this statement:

$$\int_{g(x)}^a f(t)dt$$

$$F'(x) = f(g(x)) \cdot g'(x)$$

Moreover, if we have a $g(x)$ and a $k(x)$:

$$\int_{g(x)}^{k(x)} f(x)dx = \int_a^{g(x)} f(x)dx + \int_a^{k(x)} f(x)dx.$$

4.7 Other Theorems

4.7.1 Integral Mean Value Theorem

The integral MVT is a direct consequence of the MVT for derivatives.
For a function $f(x)$, there is a point $c \in [a, b]$ for which

$$\int_a^b f(x)dx = f(c) \cdot (b - a)$$

4.8 Integral Approximation

4.8.1 Left Endpoint Sum

A left endpoint approximation multiplies the left end of each rectangle with the width of each piece for n rectangles. In other words:

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

This may also be written as

$$\Delta x \left[f(x_0) + f(x_1) + \cdots + f(x_{n-2}) + f(x_{n-1})\right]$$

The approximate error bound for this technique are given by the inequality

$$E_{LES} \leq \frac{(b-a)^2}{2n} |\max f'(x)|$$

4.8.2 Right Endpoint Sum

A right endpoint approximation is essentially the same as a left endpoint approximation; however, instead of choosing to multiply each width by the left end, we choose to multiply by the right endpoint. The sum is given as

$$\sum_{i=1}^{n} f(x_i)\Delta x = \Delta x \left[f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + f(x_n)\right]$$

The approximate error bound for a right endpoint sum is the same as it is for a left endpoint sum:

$$E_{RES} \leq \frac{(b-a)^2}{2n} |\max f'(x)|$$

4.8.3 Midpoint Sum

A midpoint sum, by comparison, takes the function evaluated at the middle of two points x_{i-1}, x_i so that the expression becomes

$$\sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right) \Delta x = \Delta x \left[f(\overline{x_1}) + f(\overline{x_2}) + \cdots + f(\overline{x_{n-1}}) + f(\overline{x_n})\right]$$

The error bound for this approximation technique is given by the formula:

$$E_M = \frac{(b-a)^3}{24n^2} |\max f''(x)|$$

4.8.4 Trapezoidal Sum

A trapezoidal sum supposes that each segment is represented by a trapezoid, rather than a rectangle, so that one edge of the shape is sloped to account for a difference in endpoint values. The sum is given by the following structure:

$$\frac{\Delta x}{2} \left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

or

$$\frac{\Delta x}{2} \left(f(x_0) + f(x_n) + 2 \sum_{i=1} n - 1 f(x_i)\right)$$

The approximate error bound for this sum is given by:

$$E_T = \frac{(b-a)^3}{12n^2} |\max f''(x)|$$

4.8.5 Simpson's Rule

Simpson's rule provides an approximation of the area under some curve with parabolas. The general prerequisite to using Simpson's rule is that the number of boxes, n, must be an even number. Otherwise, the approximation will not work. An approximation by Simpson's rule is given by:

$$\frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right]$$

Simpson's rule and the Trapezoidal and Midpoint approximations may be related by

$$S_n=\frac{1}{3}T_n+\frac{2}{3}M_n$$

The approximate error bound for Simpson's rule is given by:

$$E_S=\frac{(b-a)^5}{180n^4}|\max f^{(4)}(x)|$$

4.9 Improper Integrals and Domain Issues

The definition of a definite integral $\int_a^b f(x)dx$ requires that a, b be finite, and that f(x) be integrable on the interval. If any of the two endpoints, instead, go to infinity, we can take the limit as the particular endpoint goes to infinity. For example,

$$\begin{aligned}\int_1^\infty \frac{1}{x}dx &= \lim_{a\to\infty}\left[\ln|x|\right]_1^a\\ &= \lim_{a\to\infty}\ln a-0\\ &\to\infty\end{aligned}$$

If the limit goes to a finite number, the integral is **convergent**. Otherwise, it is classified as **divergent**. Thus, the integral shown above is divergent. If both endpoints are infinity, we can choose an arbitrary point c within the interval of the integral and use the properties of definite integrals:

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}} dx &= \int_{-\infty}^{0} \frac{e^x}{1+e^{2x}} dx + \int_{0}^{\infty} \frac{e^x}{1+e^{2x}} dx \\
&= \lim_{a\to-\infty} [\arctan e^x]_a^0 + \lim_{b\to\infty} [\arctan e^x]_0^b \\
&= \arctan 1 - \lim_{a\to-\infty} \arctan e^a + \lim_{b\to infty} \arctan e^b - \arctan 1 \\
&= \lim_{a\to-\infty} \arctan e^a + \lim_{b\to\infty} \arctan e^b \\
&= 0 + \frac{\pi}{2} \\
&= \frac{\pi}{2}
\end{aligned}$$

The same principle applies to limits with infinite discontinuities at either its endpoints or within the interval being integrated.

1. If $f(x)$ is continuous on $[a, b)$ and has a discontinuity at b, then the integral $\int_a^b f(x)dx$ is given by $\lim_{c\to b^-} \int_a^c f(x)dx$.

2. If $f(x)$ is continuous on $(a, b]$ and has a discontinuity at a, then the integral $\int_a^b f(x)dx$ is given by $\lim_{c\to a^+} \int_c^b f(x)dx$.

3. If $f(x)$ is continuous on (a, b) but has a discontinuity at both a, b, then the integral $\int_a^b f(x)dx$ may be solved by choosing an arbitrary point c within the interval and splitting the operation so that it may be solved by one of the previous statements.

4. If $f(x)$ is continuous on $[a, b]$ except for some $c \in (a, b)$ at which there exists an infinity discontinuity, then the integral may be split at the constant c and solved by either of the first two statements.

There are, in addition, several theorems that prove helpful in determining if an integral either converges or diverges.
For the integral $\int_1^\infty \frac{1}{x^P} dx$, the integral converges if $P > 1$ to $\frac{1}{P-1}$; it diverges otherwise, when $P \leq 1$.
The special integral $\int_0^\infty x^n e^{-x} dx$ converges $\forall n \in \mathbb{Z}^+$; this can be proved by induction.
In addition, for two functions $f(x), g(x)$ such that $0 \leq f(x) \leq g(x)$:

1. If $\int_a^\infty g(x)dx$ converges, then $\int_a^\infty f(x)dx$ converges.
2. Similarly, if $\int_a^\infty f(x)dx$ diverges, then $\int_a^\infty g(x)dx$ diverges.

4.10 Volume

4.10.1 Disk/Washer Method

For a solid of revolution formed by a function $f(x)$ with a hole formed by $g(x)$ revolved around some axis $y = a$, the volume of the solid is given by the formula:

$$\pi \int_a^b [r_1]^2 - [r_2]^2 dx$$

where r_1 indicates the radius formed by $f(x)$ about the axis, and r_2 indicates the radius formed by $g(x)$.
This adheres with the definition of the area of a cylinder — if we imagine that we have an infinite number of cylinders with height dx and radius $f(x)$, we are given a sum of the volumes of all of the cylinders. We may change the basis of rotation to a vertical line. In that case, the formula becomes

$$\pi \int_a^b [f(y)]^2 - [g(y)]^2 dy$$

Notice how the execution is essentially the same — only the variable we are integrating with respect to has changed.
To apply a different axis of rotation, we may apply the same logic as with the first formula.

4.10.2 Shell Method

The volume of a solid of revolution formed by a function $f(x)$ can be given by the formula:

$$2\pi \int_a^b p(x)h(x)dx,$$

with $p(x)$ representing the average distance of the shell from the axis of rotation, and $h(x)$ representing the height of the function.

4.10.3 Other Remarks

The disks/washers method is comprised of cylinders perpendicular to the axis of rotation. The shells method, however, is comprised of hollow cylinders ("shells") that are parallel to the axis of rotation.

5 Polar and Parametric Calculus

5.1 Derivatives of Parametric Equations

Given a system of parametric equations $x(t)$, $y(t)$, the derivative of the parametric equation is given by

$$\frac{y'(t)}{x'(t)}$$

which follows the general intuition that the slope of a function at a point is equal to $\frac{rise}{run}$.

5.2 Derivatives of Polar Equations

Polar equations are essentially parametric equations. We have the identities $x = r\cos\theta$ and $y = r\sin\theta$, where r denotes a specific function of θ. Therefore, the derivative of a polar equation is given by

$$\frac{\frac{d}{d\theta}(f(\theta)\sin\theta)}{\frac{d}{d\theta}(f(\theta)\cos\theta)}$$

Note: When the singular derivatives evaluate to zero, it is important to make sure of what each mean.

- When $y'(\theta) = 0$, the slope is horizontal at that point because there is no movement vertically.
- When $x'(\theta) = 0$, the slope is vertical because there is no change horizontally.
- When both pieces evaluate to 0, the derivative at that point is indeterminate.

At the pole: if $r'(\theta) \neq 0$ and $r(\theta) = 0$, then the tangent at the pole is $r = f(\theta)$.

5.3 "Area Under the Curve" of a Polar Graph

Integration in terms of area for polar graphs means taking the area of a single "slice" of a graph, were one to cut from the middle outwards like a circular cake. The area under a polar curve is given by

$$\frac{1}{2}\int_{\alpha}^{\beta}(f(\theta))^2 d\theta$$

When calculating the areas of polar graphs, it is important to consider symmetry at all times — due to difficulty with sign changes, calculating a single piece of the area and multiplying by the appropriate factor will be much easier to deal with.

5.4 Integrals of parametric functions

Given parametric functions $x = f(t)$ and $y = g(t)$, the area under the curve of the complete parametric graph is given by:

$$\int_{a}^{b} g(t)f'(t)dt$$

with $a = f(a)$ and $b = f(b)$.
This is can be derived from recalling that the area under a regular graph is given by $\int_a^b F(x)dx$. To switch out the variable of integration, we need to take a differential: $dx = f'(t)dt$. We also know that $F(x) = F(f(t)) = g(x)$; therefore, we get the aforementioned formula.

5.5 Arc Length

The arc length of a function on a closed interval $[a, b]$ is given by

$$L = \int_{a}^{b} \sqrt{1 + [f'(x)]^2} dx$$

This may be extended to parametric equations, which are expressed as

$$L = \int_{a}^{b} \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2} dt$$

The formula for polar arc length follows a similar format. For a polar function $r(\theta)$ on an interval $[\alpha, \beta]$,

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + (r')^2} d\theta$$

5.6 Surface Area

If we apply similar logic to surface area as we do with volume, then we find that the method of splitting the function into tiny cylinders does not work. Instead, we may use arclength, otherwise indicated by $\sqrt{dx^2 + dy^2} = d\ell$. What this creates for us is are tiny frustums, of which each surface area may be found with $2\pi\bar{r}d\ell$. Thus, the surface area of a function may be expressed as

$$\int_a^b 2\pi r d\ell$$

where we may replace $d\ell$ with a formula to fit needs:

- $d\ell = \sqrt{1 + (y'(x))^2} \cdot dx$
- $d\ell = \sqrt{(x'(y))^2 + 1} \cdot dy$
- $d\ell = \sqrt{(x'(t))^2 + (y'(t))^2} \cdot dt$
- $d\ell = \sqrt{r^2 + (r'(\theta))^2} \cdot d\theta$

5.6.1 Surface Area for Polar

The logic for finding the surface area of a solid of revolution in polar form lies in the parametric relationships in the graphs. Recall:

$$x = r\cos\theta, y = r\sin\theta$$

Thus, the equations for finding the surface area for a polar equation $r = f(\theta)$ become

$$A = 2\pi \int_\alpha^\beta f(\theta)\sin\theta\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}d\theta \qquad \text{(about polar axis)}$$

$$A = 2\pi \int_\alpha^\beta f(\theta)\cos\theta\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}d\theta \qquad \text{(about line } \theta = \tfrac{\pi}{2}\text{)}$$

For alternate axes of rotation, just remember that the radius is the distance from the axis to the graph, so that we may just take the difference between as our radius. For example, rotating a circle $r = 2$ around the axis $5\sec\theta$ reveals an effective radius of $5 - x = 5 - 2\cos\theta$.

5.7 Smoothness

In order to find the surface area or arc length without any significant problems, we need smoothness.
In the context of arc length, smoothness just indicates that the curve does not cross over itself at any point — if it does, then we will need to split the integral up into pieces and calculate each separately.
In the context of surface area, smoothness implies that the square root part of our differential for $d\ell$ does not equal 0 at any point. If it does, then we will need to split up the integral.

6 Differential Equations

6.1 Definitions

A differential equation is one that relates a function to one or more of its derivatives.

6.2 Integration

Some differential equations come in the form $y^{(n)} = f(x)$, for which we may just integrate both sides until we reach a solution.

6.3 Separable Differential Equations

In order to solve differential equations of a certain format, we may separate them; that is, divide one side by an expression to obtain an integrable equation with differentials. An example of a separable differential equation is $y' = yx$, since we may just separate the variables:

$$\begin{aligned} \frac{dy}{dx} &= y \cdot x \\ \frac{1}{y} \cdot dy &= x \cdot dx \\ \int \frac{1}{y} dy &= \int x dx \\ \ln|y| &= \frac{x^2}{2} + C \\ y &= e^{\frac{x^2}{2} + C} \\ y &= Ce^{\frac{x^2}{2}} \end{aligned}$$

Note: the attached constant is incredibly important to attach at the proper time. The solution shown above is a **general** solution of a differential equation. If we were to attach initial conditions to the equation (e.g. $y(0) = 5$) then the resulting equation ($y = 5e^{\frac{x^2}{2}}$) is called a **particular** solution.

6.3.1 Slope Fields

A slope field is a graphical way of representing the many different solutions of a differential equation, since finding a definite solution forone can be particularly difficult. Drawing a slope field consists of drawing a short line segment at a point that represents its specific slope.

6.3.2 Euler's Method

Euler's method is an algorithm for approximating differential equations. Given a differential equation $y' = F(x, y)$ and an initial point (x_0, y_0), one may approximate the direction of the function by taking a small "step" in the direction of the slope indicated at the current point and repeating until the desired number of points is reached. In general, a smaller step size will give you a better approximation. For each step, the algorithm computes:

$$x_{n+1} = x_n + \Delta x, y_{n+1} = y_n + \Delta F(x_n, y_n)$$

6.3.3 Exponential Growth

If y' is proportional to y, we may solve to get some sort of exponential equation:

$$\begin{aligned} y' &= ky \\ \frac{1}{y}dy &= k \cdot dx \\ \ln|y| &= kx + C \\ y &= Ce^{kx} \end{aligned}$$

6.3.4 Logistic Function

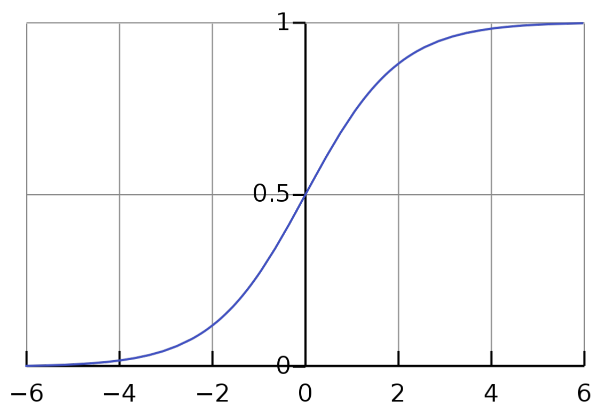
An important example of a differential equation used to model various scenarios like population growth is the logistic, or sigmoid function. The model rests on the principle that the growth rate of a population decreases as the environment's carrying capacity, K, is reached. The corresponding equation:

$$P' = bP(1 - \frac{P}{K})$$

where P denotes population, P' denotes the rate of change of the population, K denotes the carrying capacity, and b, a constant. To solve:

$$\frac{K}{P(K-P)} \cdot dP = b \cdot dt$$
$$\int \frac{K}{P(K-P)} dP = \int bdt$$
$$\int \frac{1}{P} dP + \int \frac{1}{K-P} dP = \int bdt$$
$$\ln|P| - \ln|K-P| = bt + C$$
$$\ln|\frac{K-P}{P}| = -bt + C$$
$$\frac{K-P}{P} = Ce^{-bt}$$
$$P = \frac{K}{1+Ce^{-bt}}$$

With the graph of the logistic function:



K represents the upper limit of the function: within the graph, $K = 1$.

6.3.5 Newton's Law of Cooling

If the change in temperature T is proportional to the difference between the ambient temperature of the environment with the present temperature:

$$T' = k(T_A - T)$$

We may solve this setup with a separation of variables:

$$\begin{aligned}\frac{1}{T_A - T} \cdot dT &= k \cdot dt \\ \int \frac{1}{T_A - T} dT &= \int k dt \\ -\ln|T_A - T| &= kt + C \\ \ln|T_A - T| &= -kt + C \\ T_A - T &= Ce^{-kt} \\ T &= T_A - Ce^{-kt}\end{aligned}$$

the final equation of which is Newton's law of cooling.

6.3.6 Tank Problems

Suppose there is a tank of a volume V, in gallons, with both a rate in and rate out of r, in gallons per second. In addition, there is an inflow of matter with a concentration of c kilograms per gallon, and the unknown variable is the concentration outwards, y. The change in matter concentration in this scenario may be modeled by the representation $y' = (matterin) - (matterout)$, equal to $c \cdot r - x \cdot r$.

$$y' = c \cdot r - y \cdot r$$

This scenario can be solved with a separation of variables. Note: If the rate in and rate out are different numbers, then the differential equation cannot be solved with the method outlined.

7 Sequences and Series

7.1 Sequences

7.1.1 Definitions

A **sequence** is an enumerated list of **terms**. Generally, a sequence a with n elements is denoted by a_n; every element of the list of index i is denoted a_i. A recursive sequence is a sequence in which each successive term is defined by terms before it. A famous example of this is the Fibonacci sequence. A monotonic sequence is one that is either non-increasing or non-decreasing;

that is, they may stay constant and either increase or decrease, but never both. A strictly monotonic sequence is one that only increases or decreases without staying constant at any point.

7.1.2 Limit of a sequence

The formal definition of the limit of a sequence is stated as:

$$\begin{gathered}\text{For } L \in \mathbb{R}, \\ \lim_{n\to\infty} a_n = L \text{ iff } \forall \epsilon > 0\ \exists N > 0 \\ \text{such that } n > N \text{ and } |a_n - L| < \epsilon\end{gathered}$$

An incredibly useful theorem for finding the limits of sequences is the **handy theorem**, which says that we may express a sequence as a function with which we may use tricks like L'Hopital's rule to find the limit. Formally:

$$\begin{gathered}\text{For } L \in \mathbb{R},\ f : \mathbb{R} \to \mathbb{R} \text{ s.t. } \lim_{x\to\infty} f(x) = L, \\ \text{if } \exists N \in \mathbb{N} \text{ s.t. } \forall n > N,\ n \in \mathbb{N},\ a_n = f(n) \\ \text{then } \lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = L\end{gathered}$$

With the Handy theorem, we may use different limit properties defined for a function, in addition to the squeeze theorem.

7.2 Series

7.2.1 Definition

A series is defined to be the sum of a sequence: for a sequence a_n, its infinite series would be $\sum_{n=1}^{\infty} a_n$.

7.3 Series Convergence Tests

7.3.1 Nth Term Test (NTT)

Essentially, the NTT asks for the nth term of a sequence a_n as $n \to \infty$. If $\lim_{n\to\infty} a_n = L$ for $L \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is certainly divergent. If $\lim_{n\to\infty} a_n = 0$, then the outcome of the test is uncertain; the series may or may not diverge.

7.3.2 Geometric Test

A geometric sequence follows the very specific form of $a(b)^n$, with a, b being constants. Its series, therefore, is $a\sum_{n=1}^{\infty} b^n$. A geometric sequence converges if $b \in (-1, 1)$, and diverges otherwise. This test only applies to valid geometric series. The converging value of a geometric series is given by the formula

$$\frac{a}{1-r}$$

where a is the coefficient and r is the ratio. Meanwhile, the nth term of a geometric series can be found with the formula

$$\frac{a(r^n - 1)}{r - 1}$$

7.3.3 Telescoping Test

A telescoping series consists of terms that eventually cancel each other out. Usually, this requires multiple iterations written out to realize. A typical telescoping series usually looks like

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

from which it is obvious that successive terms will cancel each other out. Some are not obvious at first glance, and may be solved for through partial fraction decomposition.

7.3.4 Integral Test

The integral test just involves taking an improper integral towards infinity of the expression of the series. If the integral diverges, then so does the sum of the sequence a_n; likewise, if the integral converges, then the series converges as well. However, using the integral test requires three prerequisites:

1. The function is continuous on the domain we care about.
2. The function decreases on the domain we care about.
3. The function is positive on the domain we care about.

The second condition can just be proven with the use of the NTFDT, while the others can be proven with just a simple observation of the function.

7.3.5 p-Series Test

A p-series takes a specific form of

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

for some constant p. By the test, the series diverges if $p \leq 1$, and converges otherwise. This test is proven with the use of the integral test.

7.3.6 Basic Comparison Test (BCT)

The BCT leverages inequalities that may be created between different functions. It leverages known properties of another series in addition to the series that is being tested; hence, its name of a "comparison test." For a series that is known to converge $\sum_{n=1}^{\infty} a_n$ compared to an unknown series $\sum_{n=1}^{\infty} b_n$: if $b_n \leq a_n$, for all n that we care about, then we know that $\sum_{n=1}^{\infty} b_n$: if $b_n \leq a_n$ converges as well.

Conversely, if $\sum_{n=1}^{\infty} a_n$ diverges and $a_n \leq b_n$, then $\sum_{n=1}^{\infty} b_n$ is known to diverge. The BCT requires that both a_n and b_n are positive and decrease on the interval we care about.

7.3.7 Limit Comparison Test (LCT)

The LCT also leverages the known properties of one other series, with the same prerequisites as the BCT. The test states that for two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ with one of the series having a known convergence/divergence property, and if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c | c \in \mathbb{R}^+$$

then either $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both convergent or divergent.

7.3.8 Alternating Series Test (AST)

For a series of the form

$$\sum_{n=0}^{\infty} (-1)^n a_n$$

or others in which the terms of the sequences alternate signs, (another example being $\sum_{n=0}^{\infty} \cos{(\pi n)} a_n$), the AST says that if:

1. $\lim_{n\to\infty} a_n = 0$

2. $a_{n+1} \leq a_n$,

then the alternating series converges.

7.3.9 Ratio Test (Ra. T)

For an infinite series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$, the ratio test says that:

1. $\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

2. $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \to \infty$

3. The test is inconclusive if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$

7.3.10 Root Test (Ro. T)

The root test has no prerequisites. The root test states that for a series $\sum_{n=1}^{\infty} a_n$: We define a value L such that $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$.

1. If $L > 1$ (including $L \to \infty$), then the series diverges.

2. If $L = 1$, then the test is inconclusive.

3. If $L < 1$, then the series converges.

7.4 Power Series

A power series is defined to be a series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

where c is the "center" about which the series exists. Effectively, the series is a function for which we can determine the interval of convergence. The interval of convergence is dictated by the radius of convergence, usually denoted R. There are two specific theorems that pertain to this goal.

7.4.1 Theorem 1

For a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$, there are three conditions of convergence, of which the series fulfills exactly one.

1. The series converges for only $x = c$, or, $R = 0$.
2. The series converges absolutely only within a balanced interval around c of radius R, or when $|x - c| < R$. The theorem does not have a conclusion at exactly $|x-c| = R$, so we must check endpoints individually.
3. The series converges absolutely everywhere.

Note that each interval always contains the center, c — a power series always converges at its center.

7.4.2 Theorem 2

Define a function f such that $f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n$ with $R > 0$. Then,

1. $f(x)$ is continuous and differentiable on $(c - R, c + R)$.
2. $f'(x) = \sum_{n=1}^{\infty} a_n \cdot n \cdot (x - c)^{n-1}$
3. $\int f(x)dx = \sum_{n=0}^{\infty} a_n(x - c)^{n+1} \cdot \frac{1}{n+1}$

Any power series and its derivatives/integrals share the same radius of convergence. However, endpoints may behave differently. Note that one should be careful about starting points when differentiating or integrating formulaically.

7.4.3 Representing Functions

One extremely useful way to think about power series is to view them as functions who behave like a corresponding function around within their interval of convergence. A most basic example of this is a manipulation with the geometric series.
With the function $\frac{1}{1-x}$, one can easily tell that it takes the form of the value to which a geometric series converges, with the common ration being x. Therefore,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, x \in (-1, 1)$$

One can manipulate functions to take a similar form to $\frac{1}{1-x}$ — take $\frac{x}{7-x^2}$, for example.

$$\frac{x}{7-x^2} = x \cdot \frac{1}{7-x^2} = \frac{x}{7} \cdot \frac{1}{1-(\frac{x^2}{7})}$$

$$\frac{x}{7-x^2} = \frac{x}{7}\sum_{n=0}^{\infty}\left(\frac{x^2}{7}\right)^n, x \in (-\sqrt{7}, \sqrt{7})$$

We can also differentiate and integrate series to achieve a desired power series representation. For the function $\ln(5-x)$, recall that it equals $-\int \frac{1}{5-x}$. Therefore,

$$\frac{1}{5-x} = -\int \frac{1}{5} \cdot \sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n$$

We can thus integrate each successive term until we find a series; in this case, $C - \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)5^{n+1}}$, where $C = \ln 5$ after plugging in $x = 0$.

7.5 Taylor and Maclaurin Series

While finding the power series representation of a function through purely the geometric series can be useful, it is also extremely limited in its applications. Thus, a more general form of approximating a function can be found through a Taylor series representation. A Taylor series is defined as:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \end{aligned}$$

As shown, the center of the power series is a constant a. A Maclaurin series is simply the Taylor series about $x = 0$.

It's observed that the series yields a polynomial function as an approximation for $f(x)$ — thus, any polynomial function will be represented perfectly as a finite Taylor series. For functions like $\cos x$, what we get from finding its Taylor series is an infinite degree polynomial, of which we can find a partial sum up to degree n to find its n^{th} degree Taylor polynomial; or,

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

7.5.1 Binomial Series

A binomial function of a form $(1+x)^k$ can be expressed as a power series that converges if $|x| < 1$:

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

This is an extension of the binomial theorem that works not only when $k \in \mathbb{N}^+$ but for all $k \in \mathbb{R}$.

7.5.2 Lagrange Error Bound

This motivates the idea of a remainder, or an error bound; the error between an n^{th} degree Taylor polynomial and the function $f(x)$ is defined to be $R_n(x) = f(x) - T_n(x)$. The error bound theorem states that:
With $f(x) = T_n(x) + R_n(x)$, $R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$ This can be restated as:

$$|R_n| \leq \frac{f^{n+1}(z)|x-c|^{n+1}}{(n+1)!}$$

where R_n is the remainder and z is some value between c and x. This can be rewritten again to become easier to use:

$$|R_n| \leq \frac{k|x-c|^{n+1}}{(n+1)!}$$

for some $k \geq f^{n+1}(z), \forall z \in (x, c)$

7.5.3 Other Remarks

Even vs. Odd
You'll notice that the Taylor series for $\cos x$ is made up of entirely even powers; hence, it is said to be an even function. The same principle applies to $\sin x$; all of its powers are odd, so it is an odd function.
Complex
Take the Taylor series representation for a function e^{ix}:

$$\sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = 1 + ix - \frac{x^2}{2} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} \cdots$$

Now notice that:

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots$$

and

$$i \sin x = i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\right)$$

Thus, we can conclude that

$$e^{ix} = \cos x + i \sin x$$

the right hand side of which is De Moivre's theorem, which represents a complex number. The above identity is known as Euler's identity, which relates exponential forms to complex numbers. This gives rise to what is dubbed the "most beautiful equation in all of mathematics":

$$e^{i\pi} + 1 = 0$$

8 Physics Applications

8.1 Work

The basic law of work for a force perpendicular to a surface states that

$$W = F \cdot x$$

or, when the force is at an angle θ to a surface,

$$W = F \cdot x \cdot \cos\theta$$

This formula works perfectly well for constant forces across the entirety of the distance, but it cannot reasonably account for variable forces; to deal with those situations, calculus is needed. As work is simply the area under a force-distance graph, work can be expressed as the area under the curve of a function of distance that determines force, or

$$W = \int_a^b F(x)dx$$

This is motivated by the idea that, with a variable force, a tiny change (ΔW) is equal to the force at a point times a tiny subinterval Δx. In other words:

$$W = \sum \Delta W = \sum F(c)\Delta x$$

When we have a relationship $\Delta W = F(x)\Delta x$, we may simply integrate to find the total W.

8.1.1 Springs

We have Hooke's law,

$$F_{spring} = k \cdot x$$

where k is the spring constant. While the whole of Hooke's law contains a negative sign to account for the vector direction of the force, the negative sign will be omitted as only the magnitude is being discussed. Since the force exerted is not constant across a certain distance, the work done must be found with an integral. If one requires a newtons of force to compress a spring b meters, then the spring constant k must be $\frac{a}{b}$; giving us the equation $F = \frac{a}{b} \cdot x$. Therefore, $\Delta W = \frac{a}{b}x \cdot \Delta x$, and

$$W = \int_{x_1}^{x_2} \frac{a}{b} \cdot x dx$$

8.1.2 Gravitation

Newton's Law of Universal Gravitation states:

$$F = G\frac{m_1 m_2}{r^2}$$

However, for a problem where the gravitational force as a whole depends on an object's distance from a body, it may be helpful to simplify the formula to an inverse square relationship with the radius:

$$F = \frac{C}{r^2}$$

From this, a relationship between work, force and distance can be created: $\Delta W = \frac{C}{x^2} \cdot \Delta x$. The same can be applied to Coulomb's law of charge ($F = k\frac{q_1 \cdot q_2}{d^2}$), as both follow the similar format.

8.1.3 Chains

The weight of a chain can be gathered from its density per unit length. Therefore, the force of lifting an increment of chain can be expressed as $\Delta F = \rho \cdot g \cdot \Delta y$. Therefore, the relationship between work done by lifting the chain some distance and an increment of length is then $\Delta W = \rho \cdot g \cdot \Delta y \cdot y$.

8.2 Centroids

For an object with mass m and a distance from the axis x, its moment is defined as $m \cdot x$. The total moment of a system is all of the individual moments summed together. The center of mass of a system on just the x axis is all of the individual moments of each component summed together, then averaged over the total mass; or,

$$\overline{x} = \frac{\sum m_n x_n}{\sum m_n}$$

The concept can be extended to two dimensions. If the moment about the y axis is M_y, and the moment about the x axis is M_x, then the centers of mass are simply

$$\overline{x} = \frac{M_y}{m}, \overline{y} = \frac{M_x}{m}$$

Notice how M_y pertains to an equation not involving the y axis, but the x axis; this is because masses on the x axis are really rotating about the y axis.

Not every system will be perfectly packaged point masses that are separate from each other; often, they will be continuous surfaces. Thus, the mass and moments must be found with integrals, or sums of tiny pieces of mass and distance over the whole. Mass is intuitive; for a planar lamina, defined with the functions $f(x)$ and $g(x)$ with a density of ρ, the mass is simply

$$m = \rho \int_{x_1}^{x_2} f(x) - g(x) dx$$

The moments, however, are a bit more complicated. After derivation, a simplified formula presents

$$M_x = \rho \int_{x_1}^{x_2} \frac{f(x) + g(x)}{2} (f(x) - g(x)) dx$$

$$M_y = \rho \int_{x_1}^{x_2} x(f(x) - g(x)) dx$$

The x and y coordinates of the lamina are then, respectively,

$$(\overline{x}, \overline{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right)$$

8.2.1 Theorem of Pappus

A useful application of knowing the center of mass of an area is the Theorem of Pappus, which gives the volume of a solid of revolution of a planar mass. For a region R with an area A revolving about a line L such that L does not pass through the interior of R, with a distance of r from the centroid of R to L, the volume is defined as:

$$V = 2\pi r A$$

8.3 Fluid Force

The pressure of a fluid with a weight density w against an object at a height h is defined as

$$P = wh$$

We also have Pascal's Principle, which states that pressure is proportional to force:

$$F = PA$$

where A is the area that the pressure is applied onto. When a mass is horizontal and has a constant height while submersed in a fluid, then no calculus is needed. However, if the mass is vertical against the fluid, then the height is not constant; therefore, we need calculus. For an object submersed in a fluid, the force of the pressure against the object is

$$F = w \int_a^b h(y)L(y)dy$$

where $h(y)$ is a function of the fluid depth and $L(y)$ is the mass's horizontal length at the mark y.

9 Appendix

9.1 Trig Identities

Pythagorean	Double Angle	Power reducing
$\sin^2\alpha+\cos^2\alpha=1$	$\sin 2\alpha=2\sin\alpha\cos\alpha$	$\sin^2\alpha=\frac{1-\cos 2\alpha}{2}$
$\sec^2\alpha-\tan^2\alpha=1$	$\cos 2\alpha=2\cos^2\alpha-1$	$\cos^2\alpha=\frac{\cos 2\alpha-1}{2}$
$\csc^2\alpha-\cot^2\alpha=1$	$\tan 2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha}$	$\tan^2\alpha=\frac{1-\cos 2\alpha}{1+\cos 2\alpha}$

Product to Sum	Sum to Product
$\sin\alpha\sin\beta=\frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$	$\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$
$\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$	$\sin\alpha-\sin\beta=2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$
$\sin\alpha\cos\beta=\frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$	$\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$
$\cos\alpha\sin\beta=\frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$	$\cos\alpha-\cos\beta=-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

9.2 Common Derivatives/Integrals

$$\int x^n dx=\frac{1}{n+1}x^{n+1}$$

$$\int \tan x dx=-\ln|\cos x|$$

$$\int e^x dx=e^x$$

$$\int \sec x dx=\ln|\sec x+\tan x|$$

$$\int \ln x dx=x\ln x-x$$

$$\int \cot x dx=\ln|\sin x|$$

$$\int \frac{1}{x}dx=\ln|x|$$

$$\int \csc x dx=\ln|\csc x-\cot x|$$

$$\int a^x dx=\frac{1}{\ln a}a^x$$

$$\int \sec^2 x dx=\tan x$$

$$\int \sin x dx=-\cos x$$

$$\int \csc^2 x dx=-\cot x$$

$$\int \cos x dx=\sin x$$

$$\int \sec x\tan x dx=\sec x$$

$$\int \csc x \cot x dx = -\csc x$$

$$\int \frac{a}{a^2 + x^2} dx = \arctan \frac{x}{a}$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin \frac{x}{a}$$

$$\int \frac{a}{x\sqrt{x^2 - a^2}} dx = \sec^{-1} \frac{x}{a}$$

$$\int \frac{a}{a^2 - x^2} dx = \frac{1}{2} \ln |\frac{x + a}{x - a}|$$

$$\int \sec x \tan x dx = \sec x$$

$$\int -\csc x \cot x dx = \csc x$$

$$\int \sinh x dx = \cosh x$$

$$\int \cosh x dx = \sinh x$$

$$\int \tanh x dx = \ln |\cosh x|$$

$$\int \coth x dx = \ln |\sinh x|$$

$$\int cschxdx = \ln |\tanh \frac{x}{2}|$$

$$\int sechxdx = \ln \arctan \sinh x$$

9.3 Polar Graphs

Circles

- $r = a \cos \theta$: Circle centered on the polar axis with diameter a, with leftmost edge at pole
- $r = a \sin \theta$: Circle centered on $\theta = \frac{\pi}{2}$ with diameter a, with lowermost edge at pole
- $r = a$: Circle centered at the pole with a radius a, diameter 2a

Limaçons ($r = a \pm b \sin \theta$**,** $r = a \pm b \cos \theta$)

- When the function is sin, the shape is centered on the line $\theta = \frac{\pi}{2}$
- When the function is cos, the shape is centered on the polar axis.
- If the operation is +, then the depressed side points down/left.
- If the operation is −, then the depressed side points up/right.
- $\frac{a}{b} < 1$: Looped Limaçon (looped on the depressed side)
- $\frac{a}{b} = 1$: Cardioid (pointed)

- $1 < \frac{a}{b} < 2$: Dimpled Limaçon (Slightly depressed side)
- $2 \leq \frac{a}{b}$: Convex Limaçon (flat side)

Roses ($r = a \sin b\theta$, $r = a \cos b\theta$)

- If b is even, then there are 2b petals.
- If b is odd, then there are b petals.
- The effective length of each petal is a.

Lemniscates ($r^2 = a^2 \sin 2\theta$, $r^2 = a^2 \cos 2\theta$)

- When the function is sin, the lemniscate is slanted, symmetric to the pole.
- When the function is cos, the lemniscate is symmetric to the polar axis and $\theta = \frac{\pi}{2}$
- The length of each "petal" in a lemniscate is equal to a.

9.4 Common Taylor Series

$$
\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
&= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\
\cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\
&= 1 - \frac{x^2}{(2!)} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \\
\sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots \\
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\
&= 1 + x + x^2 + x^3 + x^4 + \cdots \\
\ln(1+x) &= \sum_{n=1}^{\infty} (-1)^{(n+1)} \frac{x^n}{n} \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots
\end{aligned}
$$