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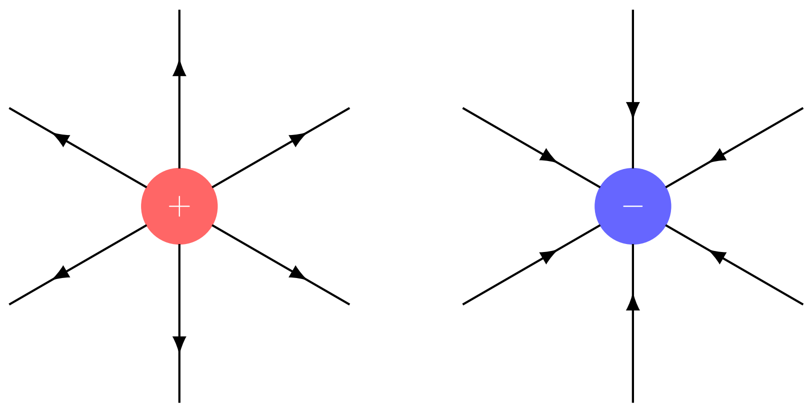
Contents

1 Electrostatics

Electrostatics focuses on the analysis of stationary charges, as opposed to *flowing* charges (which will come into play with circuit analysis). Charge is *quantized*, which means that all charges must come in multiples of e, the fundamental charge (around 1.6×10^{-19} C), as proven by Millikan's oil-drop experiment.

Charge can be either positive or negative. For an object to have a net neutral charge, the total positive and negative charges on the object must be equal in number to "cancel out."

By convention, electric field lines point *out* of positive charges and *into* negative charges.



Charge is necessarily conserved; the total charge in an isolated system will not change.

1.1 Coulomb's Law, Point Charges, and Fields

Coulomb's law for electrostatic forces states that:

$$\mathbf{F}_E = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$$

where $\frac{1}{4\pi\epsilon_0}$ is Coulomb's constant of proportionality, equal to approximately 9×10^9 Nm^2/C^2.

An electric field is an area of influence around a charged object — any charged object q in an electric field will experience a force, equal to

$$\mathbf{F} = q\mathbf{E}$$

where E is the electric field. For some source charge q_0,

$$\mathbf{E}_q(\mathbf{r}) = \frac{q_0}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$$

Note that the direction (the sign) of $\mathbf{r}$ *necessarily matters* here; it always points from the *source charge* q_0 to the *test charge* q, as the electric field quantifies the effect that the source charge has on its surroundings.

If you multiply any satellite (test) charge with $\mathbf{E}_q$, you end up with the electrostatic force $\mathbf{F}_E$ between the two charges.

Electrostatic forces follow the law of superposition; a force felt by some test charge q will be a direct sum of the forces from a distribution of source charges $q_1, q_2, \cdots, q_N$:

$$\mathbf{F}_q = \sum_{i=1}^{N} \frac{q_i q}{4\pi\epsilon_0 r_i^2}\hat{\mathbf{r}}_i$$

Consequentially, electric fields also always add at a point. The electric field due to a distribution of source charges will be

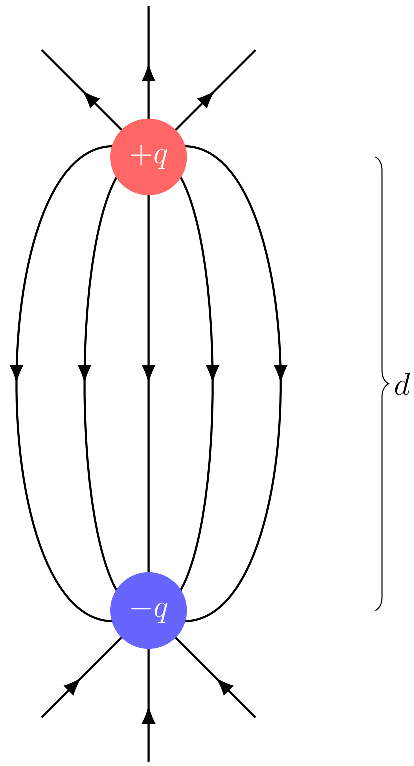
$$\mathbf{E}(\mathbf{r}) = \sum_{i=1}^{N} \frac{q_i}{4\pi\epsilon_0} \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3}$$

The electric field is *conservative*. There are several statements equivalent to this; namely, it has no curl, line integrals of the field are *path independent*, and it is the gradient of some scalar function.

$$\nabla \times \mathbf{E} = \mathbf{0}$$
$$\oint_C \mathbf{E} \cdot d\mathbf{r} = 0$$
$$\mathbf{E} = -\nabla V$$

1.2 Dipoles

A dipole is a pair of charges of equal magnitude q and opposite sign separated by some distance d.



At any point, the electric field is the sum of the electric field contributions from the positive charge $\mathbf{E}_+$ and the negative charge $\mathbf{E}_-$. Given that the dipole is centered at the origin and extends along the z axis, we have the following expression for the total field:

$$
\begin{aligned}
\mathbf{E} &= \mathbf{E}_+ + \mathbf{E}_- \\
&= \frac{q}{4\pi\epsilon_0 r_+^2}\hat{\mathbf{r}}_+ - \frac{q}{4\pi\epsilon_0 r_-^2}\hat{\mathbf{r}}_- \\
\implies \mathbf{E} &= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{(z-d/2)^2} - \frac{1}{(z+d/2)^2}\right)\hat{\mathbf{k}} \quad \{d/2 < z\} \\
\implies \mathbf{E} &= \frac{q}{4\pi\epsilon_0}\left(-\frac{1}{(z-d/2)^2} - \frac{1}{(z+d/2)^2}\right)\hat{\mathbf{k}} \quad \{d/2 < z\} \\
\implies \mathbf{E} &= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{(z+d/2)^2} - \frac{1}{(z-d/2)^2}\right)\hat{\mathbf{k}} \quad \{z < -d/2\}
\end{aligned}
$$

Notice that the field is proportional to the inverse *cube* of distance, rather than the inverse square relation we see with singular point charges.

The field produced by the dipole felt at some arbitrary point defined by polar coordinates (r, θ) is approximately

$$\begin{aligned}
\mathbf{E} &= \frac{qd}{4\pi\epsilon_0 r^3}[3\sin\theta\cos\theta\hat{\mathbf{i}} + (3\cos^2\theta - 1)\hat{\mathbf{k}}] \\
&= \frac{qd}{4\pi\epsilon_0 r^3}[2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}]
\end{aligned}$$

The qd term in the numerator is common enough that it gets its own name — it is the *electric dipole moment*. Formally, for two charges separated by some distance vector $\mathbf{d}$ (pointing from the negative to the positive charge), the electric dipole moment $\mathbf{p}$ is

$$\mathbf{p} = q\mathbf{d}$$

For a system of N charges, the net electric dipole moment is

$$\mathbf{p} = \sum_{i=1}^{N} q_i \mathbf{r}_i$$

Furthermore, if the sum of the total charges is zero, then $\mathbf{p}$ is origin-invariant. We can therefore write the above expression for the field as

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0 r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}]$$

This closely resembles a *projection* of the dipole moment $\mathbf{p}$ onto the vector delineating the point we are finding the field for.

1.2.1 Torque

In a uniform electric field, a dipole will not feel a net *force* because it holds equal and opposite charges. However, it will feel a net *torque*, as its positive end will be pulled in the direction of the field and its negative end will be pulled in the opposite direction of the field, effectively attempting to align it exactly with the direction of the field. This torque is equal to

$$\tau = \mathbf{p} \times \mathbf{E}$$

Given an orientation of the dipole displaced θ degrees from the alignment of the field, a dipole in a given configuration will have a potential energy

$$U = -pE\cos\theta = -\mathbf{p}\cdot\mathbf{E}$$

1.2.2 Multipole Expansion

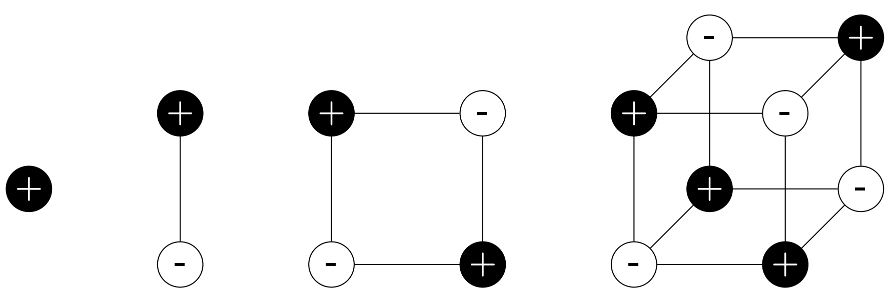
As it turns out, dipole moments are just special components of a much more generalized concept, called a *multipole expansion*. In general, it is possible to expand an expression for a field from a distribution of point charges with a higher order expansion:

$$\mathbf{E} \sim \frac{1}{r^2} + \frac{1}{r^3} + \frac{1}{r^4} + \cdots$$

The monopole moment is simply the sum of all charges:

$$Q_{tot} = \sum_{i=1}^{N} q_i$$

The r^{-2} term in the multipole expansion is proportional to this Q_{tot} due to the superposition of electric fields. The field from a dipole is proportional to the electric dipole moment, $\mathbf{p}$ — this contributes to the r^{-3} term in the expansion. We can expand this further, to quadrupole moments for the r^{-4} term, octupole moments for the r^{-5} term, and so on and so forth. This progression is analogized towards increasing *dimensionality* of the point charge distributions (though, 2-dimensional charge distributions can have nonzero quadrupole moments, and 3-dimensional charge distributions can have nonzero octupole moments).



Additionally, the lowest non-zero moment of a charge distribution is an indication towards how the strength of a field varies based on distance. Dipoles have no net monopole moment, so the highest-order surviving term is on the order of r^{-3}. Extending this further, charge configurations that have a quadrupole moment but have no net dipole moment or monopole moment will generate a field that falls on the order of r^{-4}.

1.3 Charge Distributions

Charge distributions are generalizations of point charges to continuous spaces, described by some charge density (by convention, linear charge densities are denoted λ, surface charge densities are denoted σ, and volume charge densities are denoted ρ). Charge distributions, just like point charges, describe some electric field.

$$\mathbf{E} = \frac{kq}{r^2}\hat{\mathbf{r}}$$
$$d\mathbf{E} = \frac{k \cdot dq}{r^2}\hat{\mathbf{r}}$$
$$\mathbf{E} = \int \frac{k \cdot dq}{r^2}\hat{\mathbf{r}}$$

Now, depending on context, dq can be set equal to some integrable differential. For a uniformly dense line, for example,

$$\lambda = \frac{dq}{ds}$$

$$dq = \lambda ds$$

where ds represents a tiny section of the line.

It is often the case that the field contributions $d\mathbf{E}$ do not all point in the same direction, so the individual vectors $\hat{\mathbf{r}}$ must be kept under consideration to arrive at a result that makes sense. In some cases, symmetry can be leveraged to cancel out specific components of the $d\mathbf{E}$ contributions, such that we only have to worry about the field in one direction (as opposed to all of $\mathbb{R}^3$).

1.4 Flux and Gauss' Law

For the purposes of intuition, the concept of flux can be expressed as "stuff detected over a certain area." In electrostatics, this "stuff" would be the electric field itself, and the area would be infinitesimally small pieces of our some surface we are measuring charge with.

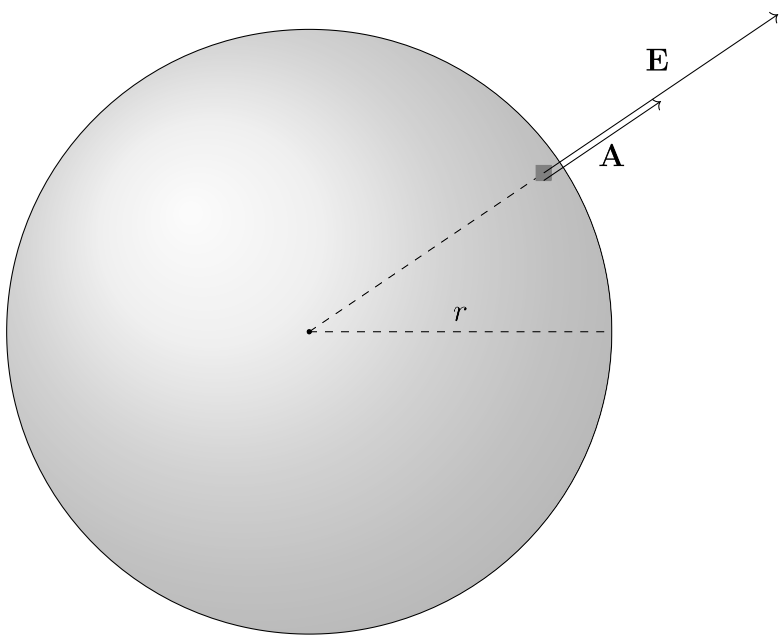
More formally, let $\mathbf{E}_j$ the electric field evaluated at some point within a patch of area denoted j. We call $\mathbf{A}_j$ the area vector for the patch at j, which is normal to the plane-like approximation of the patch with a magnitude proportional to the area of the patch. The scalar product (or the dot product) of $\mathbf{E}_j \cdot \mathbf{A}_j$ is known as the flux through the patch of the surface.

If $\mathbf{a}_j$ is orthogonal to $\mathbf{E}_j$, the dot product is zero — for a surface laying flat, a vector field will pass over it without interacting with it in any way, and it will thus detect nothing. Similarly, if $\mathbf{A}_j$ is parallel to $\mathbf{E}_j$, we will be able to detect the maximum amount of stuff passing over the patch of area.

For an entire surface, the addition of these bits of flux can be computed with a surface integral.

$$\Phi = \int_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{A}$$

For example, take a sphere:



Because the $\mathbf{E}$ and $\mathbf{A}$ vectors align with each other, the flux simply becomes

$$\Phi = E \cdot (\text{total area}) = \frac{q}{4\pi\epsilon_0 r^2} \cdot 4\pi r^2 = \frac{q}{\epsilon_0}$$

The flux seems to be independent of the volume of the sphere. In fact, this is the case for all shapes known as *Gaussian surfaces*, due to the fact that the electric field follows an inverse-square law.

Therefore, one form of Gauss' law is

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{q_{enc}}{\epsilon_0}$$

where q_{enc} refers to the total amount of charge enclosed inside of the Gaussian surface. Note that

$$q_{enc} = \int \rho dV$$

Gauss' law can also be put in *differential form*:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

otherwise expressed as

$$\nabla \cdot (-\nabla V) = -\nabla^2 V = \frac{\rho}{\epsilon_0}$$

The above is referred to as *Poisson's Equation*. When the charge density ρ is zero, we arrive at *Laplace's equation*:

$$\nabla^2 V = 0$$

Potential functions V that satisfy Laplace's equation are known as *harmonic* functions.

1.4.1 Gaussian Surfaces

There are three different convenient symmetries to utilize when applying Gauss' law.

Spheres

Spheres are most convenient for point charges and spherical objects. For a spherical Gaussian surface of radius r with applicable symmetry,

$$\oiint \mathbf{E} \cdot d\mathbf{A} = E(4\pi r^2) = \frac{Q_{enc}}{\epsilon_0}$$

Cylinders

Cylinders work well for infinite wires of charge. We may pick some arbitrary length of the wire L to calculate the field over, and additionally argue that the caps of the cylinder feel zero flux from the electric field. For a cylindrical Gaussian surface with a base radius of r,

$$\oiint \mathbf{E} \cdot d\mathbf{A} = E(2\pi r L) = \frac{Q_{enc}}{\epsilon_0}$$

Pillboxes

A "pillbox" Gaussian surface can be applied for an infinite sheet of charge. We may pick an arbitrary prism of cross sectional area A, and argue that no flux is felt through the sides of the prism that are not its base. We find the following expression:

$$\oiint \mathbf{E} \cdot d\mathbf{A} = 2AE = \frac{Q_{enc}}{\epsilon_0}$$

1.5 Shell Theorems

Recall Newton's shell theorems for the strength of gravitational fields:

1. A spherically symmetric body affects objects outside of it as if its mass were concentrated at its center.

2. The gravitational field inside of a spherically symmetric body is equal to 0.

However, these theorems are applicable generally to all vector fields that follow the inverse square law. Therefore, they apply to electric fields as well. For a hollow, spherically symmetric shell with a specific surface charge distribution:

1. The electrostatic force felt by a test particle outside of the shell is the same as if the charge of the shell were concentrated at a point particle at its center.

2. The electrostatic force felt by a test particle inside of the shell is 0.

1.6 Electric Potential

The electric potential (also known as the voltage) is a scalar quantity that describes the intensity, in a way, of the electric field at some test point. Because scalars are easier to work with than vector quantities, the electric potential is often easier to work with than the electric field itself.

We can think of the electric potential as the amount of potential energy "felt" by a test charge in the presence of another point charge or charge distribution. As an analogue to classical mechanics, the electric potential can be thought of as a "height difference" between two points in an electric field.

As a side note, we typically think of more positive charges as "high ground," and negative charges as "low ground." Thinking of it this way, positive charges will naturally go "down" the electric field, while negative charges will go "up." These signs should be kept in careful consideration when working with electric potential.

The change in electric potential ΔV as a particle moves from one point to another within an electric field is given by the line integral

$$\Delta V = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{s}$$

Note the relation between ΔV and work — work is defined as a line integral of the electric *force*. Electric potential, then, is the work per unit charge required to move a charge from some reference point to another specific point within a field. When we talk about "the potential" of a charge distribution, the aforementioned reference point is at an infinite distance away from the source of the field, at which the electric potential is 0.

A useful feature of an electrostatic field is that it follows the inverse square law, which, by extension, means that it is a *conservative field*. Therefore, it does not matter what path we take to get a charge from P_1 to P_2, because all line integrals are *path-independent* in conservative fields, by the Fundamental Theorem of Path Integration.

By extension,

$$\oint \mathbf{E} \cdot d\mathbf{s} = 0$$

Between point charges,

$$V = \frac{q}{4\pi\epsilon_0 r}$$

Also of note is that $W = Vq$. Voltage is measured in units of *volts*:

$$1 \text{ volt} = \frac{1 \text{ joule}}{1 \text{ coulomb}}$$

For a uniform field,

$$\Delta V = Ed, \ E = \frac{V}{d}$$

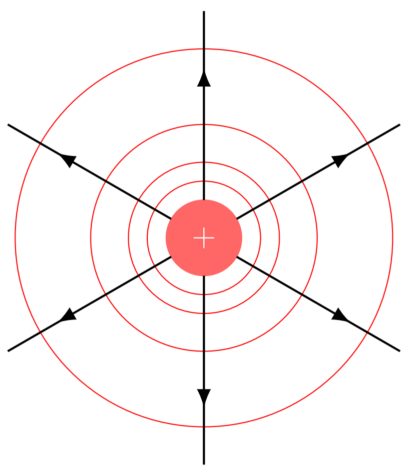
Recall the relationship between potential and work. From that relationship, we can derive the relationship between the potential energy and the potential (not to be confused with each other):

$$U = \frac{1}{2}\int \rho V \, d\tau$$

($d\tau$ refers to the volume differential, denoted d^3r by Prof. Lang).

1.6.1 Equipotentials

An *equipotential surface* is a set that contains all points in space at the same potential. This is equivalent to a level curve of a multivariable function (which, the potential function can be expressed as by holding the initial point P_1 in place and varying P_2).



The red lines indicate the equipotential lines around this point charge. Notice that these equipotentials are orthogonal to the electric field lines. Recall that the level curves of a multivariable function are normal to its gradient. This leads us into another important idea about the electric potential:

$$\mathbf{E} = -\nabla V$$

In general, equipotential lines that are spaced closer together indicate a "steeper" potential drop, and vice versa for equipotentials spaced further apart.

1.7 Energy

Because electric forces form an inverse-square vector field, electric forces are conservative — energy is not lost when charges are pushed around in electric fields. In other words, all actions are reversible. This is a direct consequence of the fact that the electric field itself is conservative, or curl-free.

The amount of work it takes to bring some charge q_1 towards another q_2 at some distance of r_{12} is

$$\begin{aligned} W &= \int \mathbf{F} \cdot d\mathbf{r} \\ &= \int_{\infty}^{r_{12}} \left(-\frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \right) dr \\ &= \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{12}} \end{aligned}$$

Again, because the electric field is conservative, it doesn't matter what path we make q_1 approach q_2 by; the work done is the same.

$$U = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_1 q_2}{r}$$

1.7.1 Energy Density

The idea of energy gets slightly weirder when we think about continuous charge distributions with infinitesimal charges at each point. It is helpful to think about an *energy density*, or the energy stored in our field at some point:

$$u = \frac{\epsilon_0}{2} E^2$$

Therefore, for an electric field, the potential energy is equal to

$$U = \int_{\text{all of space}} u d\tau = \frac{\epsilon_0}{2} \int E^2 d\tau$$

Note that $E^2 = \mathbf{E} \cdot \mathbf{E}$.

1.8 Conductors

Conductors are materials in which charges are able to move around freely. Theoretical conductors must obey the following properties:

1. The field inside $\mathbf{E} = \mathbf{0}$.
2. All of the charge on a conductor lies at its surface.
3. The *surface* of a given conductor is an equipotential — that is, all points on the surface of a conductor must be at the same potential.
4. The field outside of a conductor is locally orthogonal to the surface of the conductor.
5. The local field just outside of a conductor is $\mathbf{E}_{out} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$
6. The electrostatic force per area, or the pressure, on a conductor is $\frac{\sigma}{2}\mathbf{E}_{avg} = \frac{\sigma^2}{2\epsilon_0}\hat{\mathbf{n}}$

1.8.1 Electrostatic Shielding

An interesting consequence of the properties above: for a given solid conductor with a cavity inside, the *inside* of the cavity will be unaffected by any electrostatic field outside of the conductor. A field generated inside of the cavity, similarly, will seemingly be "subsumed" by the conductor from the outside, and it will behave as if the conductor itself is generating that field. This effect is most notable in what are called Faraday cages, which are usually used to protect people from strong electric currents and sudden electrical discharge.

1.9 Method of Images

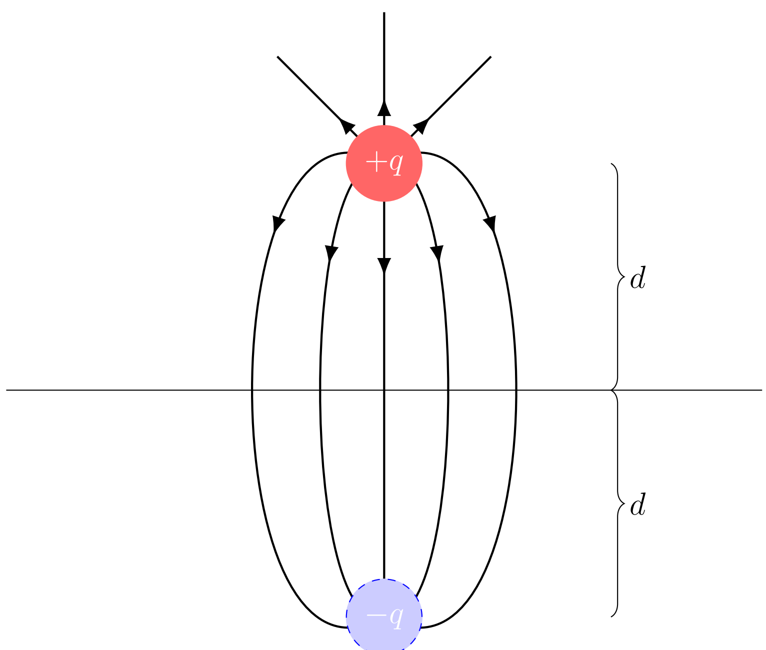
1.9.1 Uniqueness Theorem

Recall Laplace's equation:

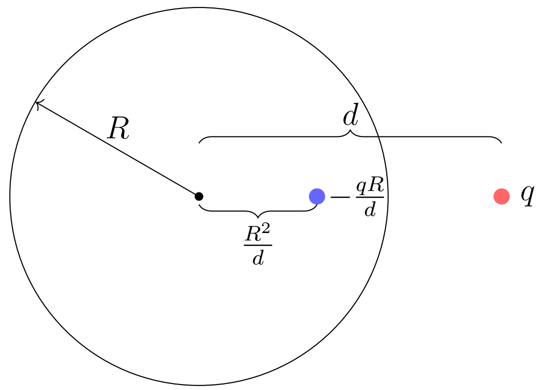
$$\nabla^2 V = 0$$

The Uniqueness theorem says that a solution to Laplace's equation in some volume $\mathcal{V}$ is uniquely determined if the potential V is determined on the bounding surface of $\mathcal{V}$. This theorem is advantageous when we wish to find the field of a complex distribution of charge in a specified region of space (and we don't care about what happens outside of the boundary).

For example, take a charge $+q$ situated a distance d above an infinite, grounded conducting plane. We want to find the electric field above the conducting plane. This situation would be hard to analyze with just electrostatic laws; instead, since we only care about one half of space (demarcated by the conducting plane), and since we know that electric field lines must be *orthogonal* to the plane, we "extend" the magnetic field by adding a negative *image* charge $-q$ a distance d below the plane. Since the boundary voltages ($V = 0$) match, the upper half of space described by this dipole situation must in turn describe the electric field of the configuration (for that region of space only), by the uniqueness theorem.



With a grounded spherical conductor, the image charge looks slightly different:

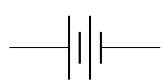


2 DC Circuits

2.1 Components

2.1.1 Battery

In a circuit, batteries serve as the providers of voltage (most of the time, through chemical potential). This V may also be referred to as $\mathcal{E}$, otherwise known as the Electromotive Force (EMF), though it should be noted that these two concepts are subtly different. In circuit diagrams, they are represented as



Where the long lines indicate the cathode, and the shorter lines indicate an anode.

2.1.2 Capacitors

A capacitor is composed of two conductors that are separated by some distance d (commonly, they are envisioned as two parallel plates). In circuits, they are components that are used to store charge. The *capacitance* of a capacitor refers to its ability to store charge, usually measured as

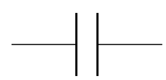
$$C = \frac{Q}{V}$$

There is also a shape-based argument, which says that

$$C = \frac{\epsilon_0 A}{d}$$

where A is the surface area of the capacitor plate, while d is the distance of separation between the two plates. Capacitance is measured in Farads (F).

In a circuit, a capacitor is typically represented by two parallel lines:



The energy of a capacitor is found through

$$E = \frac{1}{2}CV^2 = \frac{1}{2}QV$$

Capacitors in parallel form an equivalent capacitor of capacitance

$$C_{eq} = C_1 + C_2 + \cdots + C_n$$

while capacitors in series form an equivalent capacitor of capacitance

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}$$

Capacitors work by having two oppositely charged plates that attract charge to their surfaces. For example, a positively charged plate will attract electrons to the surface of the negatively charged plate, keeping them there and effectively “storing” them. Because capacitance is inversely proportional to distance, capacitor plates that are closer together will be able to store more charge. However, once the plates touch, they will be unable to store any charge, and will function as a simple conductor.

Dielectrics

In order to disallow capacitor plates from touching each other, manufacturers will typically insert a poorly conducting material, called a *dielectric*, in between the plates to keep them separate.

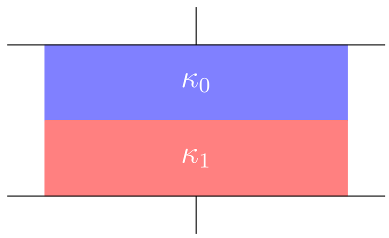
Depending on the material it is made out of, each individual dielectric will come with a corresponding *dielectric constant* (also known as the *relative permittivity*), denoted κ. This is multiplied by ϵ_0 in the numerator of the shape-based capacitance argument to find the capacitance of a capacitor using that dielectric.

$$C = \frac{k\epsilon_0 A}{d}$$

t

We can "combine" multiple dielectrics with different constants $\kappa_0, \kappa_1, \ldots, \kappa_n$ to form an equivalent κ_{eq}.

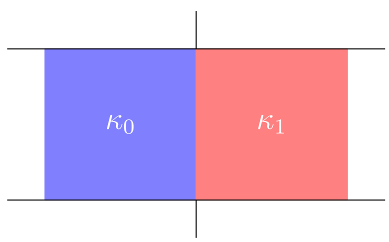
In the "layered" case,



we can think about this as a system of two capacitors in series, with one of each containing each of the different dielectrics. The equivalent dielectric constant, then, is

$$\frac{1}{k_{eq}} = \frac{1}{k_0} + \frac{1}{k_1} + \cdots + \frac{1}{k_n}$$

Dielectrics can be combined in different orientations. As another example,



We can treat this as a system of two capacitors in parallel, yielding an equivalent dielectric constant of

$$k_{eq} = k_0 + k_1 + \cdots + k_n$$

2.1.3 Resistors

A resistor is an electrical component that generates "resistance" the flow of current. Resistance, measured in ohms (Ω), quantifies the way an object or a circuit component opposes current.

The resistance of a given material can be calculated through

$$R = \frac{\rho L}{A}$$

where L refers to the length of the material (i.e. the length *along which* current will flow) and A refers to its cross sectional area. Here, ρ is the *resistivity* of a material, which is an intensive property and depends on the composition and type of material it is.

The inverse of the resistivity, $1/\rho$, is known as the *conductivity*; similarly, it measures how easily current can pass through it.

In a circuit, resistors are typically represented by a zig-zag:

Resistors in *parallel* add harmonically, while resistors in *series* add linearly (note that this is the opposite to how we calculated equivalent capacitance). For resistors in series,

$$R_{eq} = R_1 + R_2 + \cdots + R_n$$

For resistors in parallel,

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n}$$

Resistors obey Ohm's law:

$$V = IR$$

2.2 EMF

A power source, often a battery, will provide some charge-pushing power to a circuit, either through mechanical, magnetic, or chemical means. Formally, EMF is defined as the work per unit charge done by some force in a conductor, or the closed path integral around the circuit of a quantity termed $\mathbf{f}$, the force per unit charge.

$$\mathcal{E} = \oint_{\text{circuit}} \mathbf{f} \cdot d\mathbf{l}$$

This quantity $\mathbf{f}$ has some similar notion to the electric field that circulates around the circuit. We imagine that the source supplying the pushing power has a particular force per unit charge it generates, $\mathbf{f}_s$ – thus,

$$\mathbf{f} = \mathbf{f}_s + \mathbf{E}$$

In certain conditions, the closed loop integral of $\mathbf{E}$ will be 0. However, this result changes in the presence of a changing magnetic field. Without those conditions, however,

$$\mathcal{E} = \oint_{\text{circuit}} \mathbf{f} \cdot d\mathbf{l} = \int_{\text{battery}} \mathbf{f}_s \cdot d\mathbf{l} + \oint \mathbf{E} \cdot d\mathbf{l} = \int_{\text{battery}} \mathbf{f}_s \cdot d\mathbf{l}$$

2.3 Current

An electric current is the motion of charge through some medium. In many cases, this appears in the form of electrical wires transmitting charge through charge carriers, like electrons. More tangibly, the electric current through a wire is defined as the amount of charge passing some fixed mark over a unit of time. The SI unit for current is the *ampere*:

$$1\text{ A} = \frac{1\text{ C}}{1\text{ s}}$$

When working with circuits, then,

$$I = \frac{Q}{t}$$

However, when we extend this to two dimensions, we begin talking about $\mathbf{J}$, or *current density*, which is current flow over area. Conversely, current is the surface integral of the current density over the surface in question.

$$I = \int \mathbf{J} \cdot d\mathbf{A}$$

The relationship between current density and the electric field that causes or creates the current density is

$$\mathbf{J} = \sigma \mathbf{E} = \frac{1}{\rho}\mathbf{E}$$

where σ refers to the *conductivity* of a material.

We note that, in traditional understandings of circuit systems, we think about *conventional current*, which envision *protons* flowing from positive to negative (for example, the positive terminal of a battery to the negative terminal). However, in reality, *electrons* flow from negative to positive. In either case, the net flow of *charge* is the same.

2.3.1 Continuity Equation

The continuity equation is an expression of charge conservation. For a given closed surface, the current density $\mathbf{J}$ most obey

$$\oint\!\!\!\oint \mathbf{J} \cdot d\mathbf{A} = -\frac{dQ_{enc}}{dt}$$

For a steady current, this is equivalent to saying

$$\oint\!\!\!\oint \mathbf{J} \cdot d\mathbf{A} = 0$$

as there should be no charge pileup in a perfectly steady current configuration. In differential form,

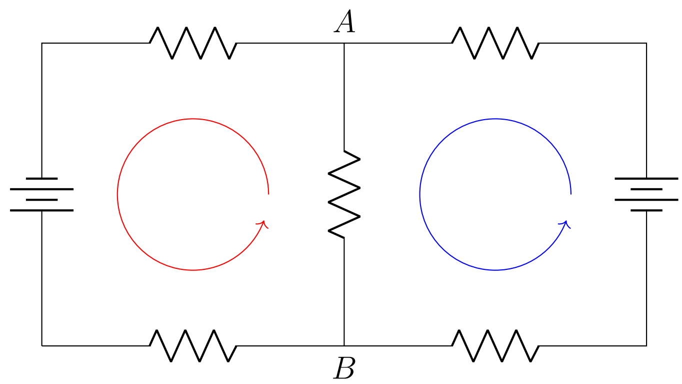
$$\nabla \cdot \mathbf{J} = -\frac{d\rho}{dt}$$

Once again, for a steady current,

$$\nabla \cdot \mathbf{J} = 0$$

2.4 Kirchhoff's Laws

When dealing with simple circuits concerning just batteries and resistors, it is helpful to use Kirchhoff's circuit laws to draw equivalencies and determine current or potential difference.



2.4.1 Junction Rule (KCL)

For any junction in an electrical circuit, the sum of currents flowing into the node and the sum of currents flowing out of the node must be equal. For the circuit above, these "nodes" are points A and B.

At A, the current splits into two branches – one that goes through the middle wire, and another that goes all the way through to the right. However, at point B, they must recombine into the same original current I_0.

2.4.2 Loop Rule (KVL)

The signed sum of potential differences throughout any closed loop must be 0.

$$\sum \Delta V_i = 0$$

For the resistors in the above circuit, we can use Ohm's law to determine these voltage drops:

$$V = IR$$

As seen in the red and blue loops, we must add up each of the potential drops and find Is such that they fulfill Kirchhoff's loop rule. There are a couple of rules for finding the signs of these voltage drops:

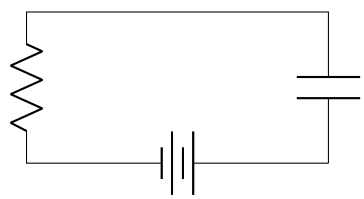
1. Going through a resistor with a current: $\Delta V_i = -IR$
2. Going through a resistor against a current: $\Delta V_i = IR$
3. Going through a battery from - to +: $\Delta V_i = \mathcal{E}$
4. Going through a battery from + to -: $\Delta V_i = -\mathcal{E}$

Using these laws, we can obtain a system of equations for the total currents in the circuit. Note that the number of equations we have will depend on the number of loops that we have, and therefore, the number of unknown currents we must solve for.

2.5 RC Circuits

RC circuits are composed of resistors and capacitors. We can solve RC circuits using the rules we currently have (namely, Kirchhoff's loop rule). However, capacitors introduce a factor of complexity into our approach, and we must now consider time-dependency.

Consider a simple RC circuit:



There is only one loop within this circuit. Furthermore, recall that

$$V = IR$$

and

$$V = \frac{q}{C}$$

Using Kirchhoff's loop rule, we would expect that

$$\mathcal{E} - \frac{q}{C} - IR = 0$$

However, we know that $I = \frac{dq}{dt}$. This gives us the differential equation

$$\frac{dq}{dt}R = \mathcal{E} - \frac{q}{C}$$

This equation is first-order separable. After setting the initial condition $q(0) = 0$, solving the IVP yields the solution

$$q(t) = \mathcal{E}C(1 - e^{-\frac{t}{RC}})$$

Furthermore,

$$I(t) = \frac{dq}{dt} = \frac{\mathcal{E}}{R}e^{-\frac{t}{RC}}$$

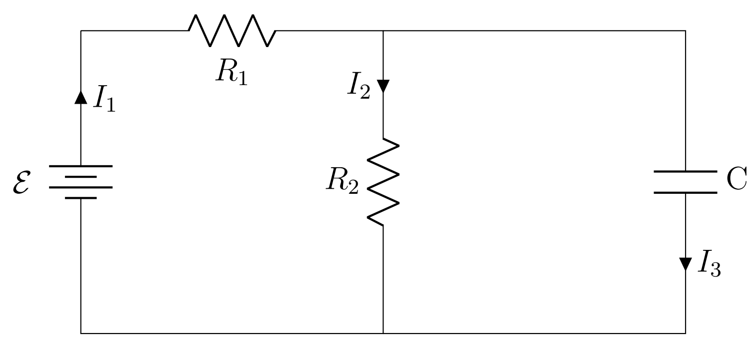
In fact, the number RC shows up so often that it is given a separate name: the time constant, also known as τ.

$$\tau = RC$$

τ gives us a nice baseline on which we can quantify the evolution of the circuit.

2.5.1 Parallel Capacitor-Resistor Systems

Suppose we have a system like the following:



When solving, we should keep in mind two limiting conditions. At $t = 0$, the capacitor, initially uncharged, will function effectively as just a wire; therefore, all of the current will try to flow through that second branch, and momentarily,

$$I_1 = I_3$$

After a very long time (say, $t = \infty$), current will be unable to flow through the capacitor branch, causing the condition

$$I_1 = I_2$$

So we know what happens at the beginning and the end of the circuit's lifetime.

What happens in between? We can start with Kirchhoff's junction law, which states that

$$I_1 = I_2 + I_3$$

With Kirchhoff's loop rule, we gather that

$$\mathcal{E} - I_1R_1 - I_2R_2 = 0$$
$$\mathcal{E} - I_1R_1 - \frac{Q_3}{C} = 0$$

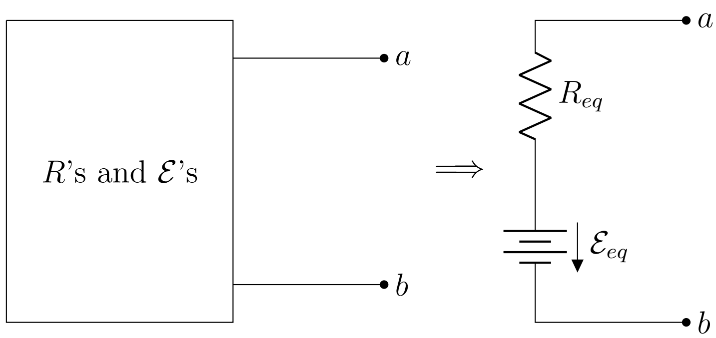
where Q_3 just refers to the charge on the capacitor. We can then draw the equivalence

$$\frac{Q_3}{C} = I_2R_2$$

Recalling that $I_k = \frac{dQ_k}{dt}$, we now have a coupled system of differential equations, which we may solve to find the solutions to I_1, I_2, I_3.

2.6 Thevenin Equivalence

Thevenin's theorem states that any complicated network of various resistances and voltages can essentially be "blackboxed" into a system with a single resistor R_{eq} and voltage source $\mathcal{E}_{eq}$. This greatly simplifies the analysis of a possibly complicated circuit with multiple branches.



Often times, we are trying to analyze the current through some load R_L or other complex circuit (for example, one involving a capacitor) to the two outgoing terminals of the blackboxed circuit (i.e. we are trying to find the Thevenin equivalent current I_{eq} between terminals a and b). To do this, we need to first find $\mathcal{E}_{eq}$ and R_{eq}, and then calculate

$$I_{eq} = \frac{\mathcal{E}_{eq}}{R_{eq}}$$

To find R_{eq}, we "short" all of the $\mathcal{E}$'s by temporarily replacing them with wires and using series/parallel rules to find the equivalent resistance. To find $\mathcal{E}_{eq}$, we short the terminals (i.e. pretend that anything past the terminals is not there) and solve for the voltage of a parallel branch.

3 Magnetostatics

Magnetostatics concerns situations in which current is *uniform* and there is no changing magnetic field.

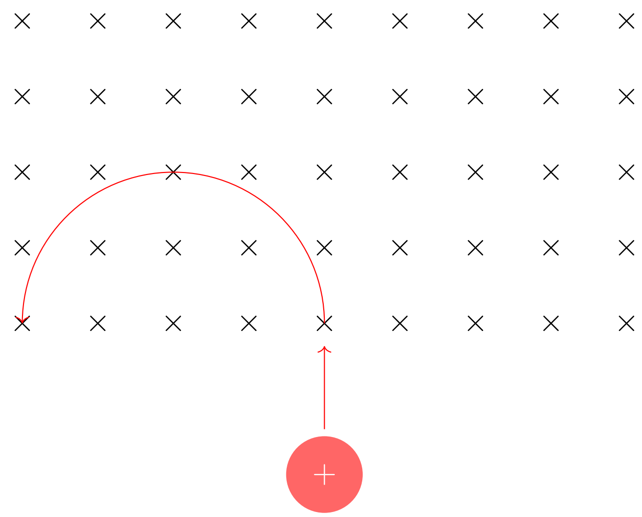
3.1 Force

The force on a moving, charged particle is summarized by the Lorentz force, which states:

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$$

$q\mathbf{E}$ is the force on the particle as affected by the surrounding electric field, commonly denoted $\mathbf{F}_E$. Similarly, the force on a particle as affected by the surrounding magnetic field is $\mathbf{F}_B = q\mathbf{v} \times \mathbf{B}$. Because the magnetic force is dependent on the particle's velocity, magnetism only affects *moving* charges.

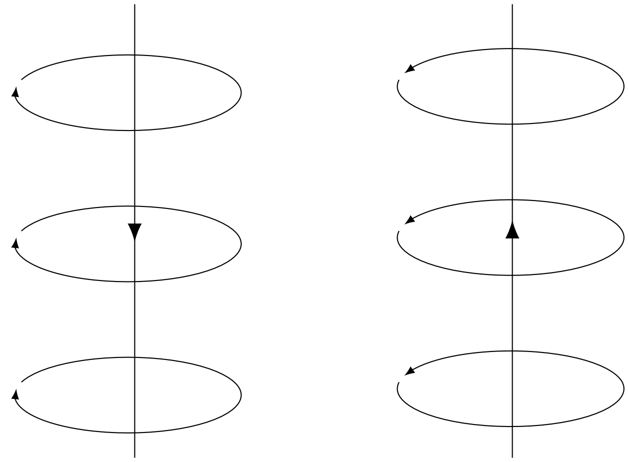
Magnetic fields generally cannot do any work on a singular particle moving across them; any component of a particle's velocity parallel to $\mathbf{B}$ will have no effect on the net force, and there will be no parallel component of the force to $\mathbf{B}$ by definition of the cross product. Therefore, magnetic fields do not actually change to magnitude of the acceleration of the particle; rather, they change the acceleration in terms of direction. In a uniform field, a charge will experience a circular deflection due to the magnetic field.



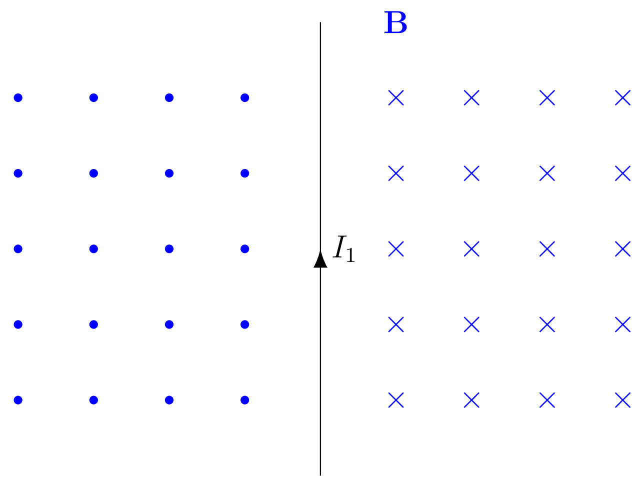
The direction in which the particle deflects can be found using the right hand rule for cross products.

3.1.1 Current Carrying Wires

Wires carrying current generate magnetic fields, which affect objects surrounding them; the orientation in which **B** will loop around the wire can be found via the right hand rule.



Two current carrying wires in the same vicinity will cause each other will to deflect. Wires with current pointing in the *same* direction will attract each other, whereas wires with current pointing in the opposite direction will repel. To see why, consider some wire



where the blue markings represent the field going into and out of the page. Another given wire to the right of the wire in the diagram will, by the right hand rule, experience a force directed *left* if its current is going in the same upwards direction, or it will experience a force directed *right* if its current is going in the opposite direction.

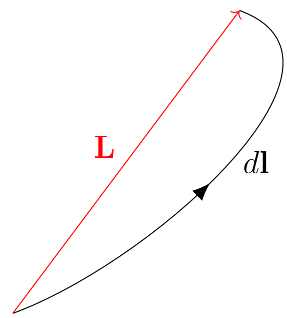
We can reframe the Lorentz force law to think about the movement of charge confined to a wire.

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} = I\mathbf{l} \times \mathbf{B}$$

For wires following some curve from point A to B with a uniform current I, we can calculate the force exerted on it from the magnetic field as

$$\int_A^B I d\mathbf{l} \times \mathbf{B} = I \int_A^B d\mathbf{l} \times \mathbf{B} = I\mathbf{L} \times \mathbf{B}$$

where $\mathbf{L}$ is the *displacement vector* from the endpoints of the wire. In other words, the only thing that really matters with respect to the force exerted on its wire is its total displacement.



3.1.2 Dipoles

Given what was stated about forces on current carrying wires in the last section, we observe that *current loops* have no net displacement, and therefore, experience no net force from a magnetic field. However, they experience a torque equal to

$$\tau = I\mathbf{A} \times \mathbf{B}$$

where **A** is the orthogonal area vector correspondign to the surface spanned by the loop. Furthermore, we define the *magnetic dipole moment*

$$\mu = I\mathbf{A}$$

(analagous to the electrostatic dipole moment **p**) such that

$$\tau = \mu \times \mathbf{B}$$

Furthermore, the potential energy held by a loop of current at some angle to **B** is

$$U = -\mu \cdot \mathbf{B}$$

In the limit where the current loop is extremely small (that is, approaches an "ideal magnetic dipole"), we can say that

$$\begin{aligned} \mathbf{F} &= -\nabla U \\ &= -\nabla(-\mu \cdot \mathbf{B}) \\ &= \nabla(\mu \cdot \mathbf{B}) \end{aligned}$$

3.2 Biot-Savart Law

The Biot-Savart law is essentially the magnetic analogue to Coulomb's law for electrostatics. It gives the incremental contribution to the field **B** from a segment of wire s as:

$$\mathrm{d}\mathbf{B} = \frac{\mu_0}{4\pi} \frac{I \ \mathrm{d}\mathbf{l} \times \mathbf{r}}{r^3}$$

where μ_0 is the *permittivity of free space*, a constant relating how magnetic fields form in a vacuum. In scalar form, this equation becomes

$$dB = \frac{\mu_0}{4\pi} \frac{I \ dl \ \sin\theta}{r^2}$$

To use this law, we are necessarily assuming that current is uniform.

3.3 Ampere's Law

Just as the Biot-Savart law is an analogue to Coulomb's law, we have Ampere's law, which is the magnetic analogue to Gauss' law. It states that

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc}$$

The line integral of the magnetic field around the closed path must always equal the constant μ_0 times the instantaneous current in the space enclosed by the path.

Note that this version of Ampere's law holds only when the continuity equation is set to 0 and we have a uniform current density; that is,

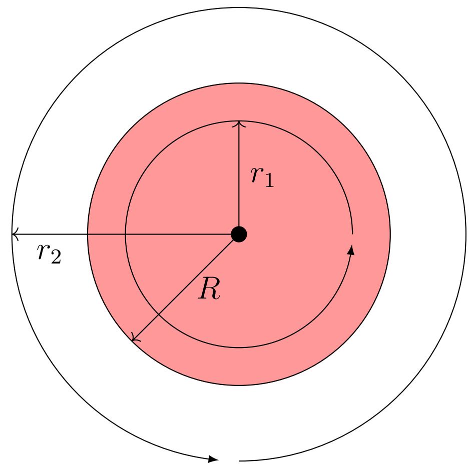
$$\nabla \cdot \mathbf{J} = 0$$

When instead $\nabla \cdot \mathbf{J} = -\frac{d\rho}{dt}$, we must consider a *displacement current* as well, and we are no longer dealing with magnetostatic cases.

3.3.1 Symmetries

Cylinders

Given a cylinder with some cross sectional current density $\mathbf{J}(r)$, we can find the circulation of the magnetic field around some circular loop a radius r from its center.



For an Amperian loop taken inside of the cylinder, we are only enclosing some section of current.

$$\oint \mathbf{B} \cdot d\mathbf{l} = 2\pi r_1 B = \mu_0 \iint \mathbf{J} \cdot d\mathbf{A}$$

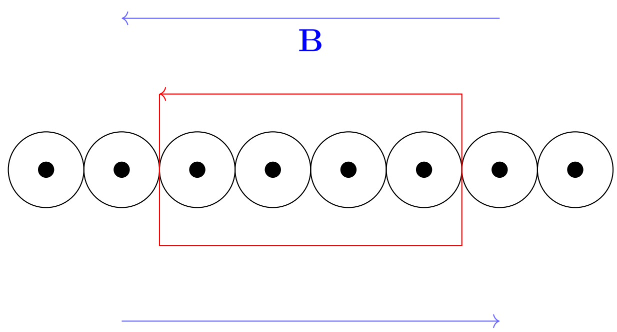
where the double integral is taken in cylindrical coordinates up to r_1.

Outside of the cylinder, we are enclosing the current in its entirety.

$$2\pi r_2 B = \mu_0 I_{total}$$

Sheets and slabs

Suppose we have a sheet of current going "out of" the page with current density per length **K**.



If we draw a rectangular loop of length L around the cross section of the sheet,

$$\oint \mathbf{B} \cdot d\mathbf{A} = B \cdot 2L = \mu_0 K L$$

and therefore,

$$B = \frac{\mu_0 K}{2}$$

with direction found through

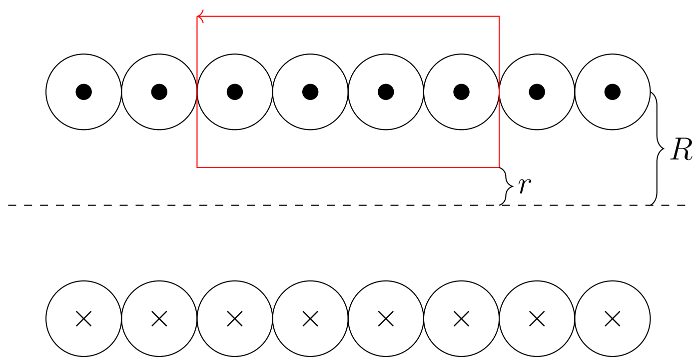
Solenoids

By using Ampere's law, we can prove that the magnetic field generated (on the inside) by a solenoid (a coil of wire) is

$$B = \mu_0(IN)$$

where N is the number of coils in the wire. The field is **0** outside of the solenoid.

To find this, consider a "closeup" on one side of a solenoid with N turns per length.



If the side length of the rectangular loop is L,

$$\oint \mathbf{B} \cdot d\mathbf{l} = BL = \mu_0 I_{enc} = \mu_0 NIL$$

And therefore, we see that

$$\mathbf{B} = \mu_0 IN\hat{\mathbf{k}}$$

3.4 Magnetic Vector Potential

The magnetic field does not have a *scalar* potential in the same way that the electric field does. Instead, we define something known as the *magnetic vector potential* **A**:

$$\mathbf{B} = \nabla \times \mathbf{A}$$

We note that the choice of $\mathbf{A}$ is not necessarily unique for each $\mathbf{B}$. We have the notion of a *gauge choice* for $\mathbf{A}$, which means that we simply choose an $\mathbf{A}$ that satisfies:

$$\nabla \times \mathbf{A} = \mathbf{B}$$
$$\nabla \cdot \mathbf{A} = 0$$

Furthermore,

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

This requires the *vector Laplacian*, which, implicitly, says

$$\nabla^2 A_x = -\mu_0 J_x$$
$$\nabla^2 A_y = -\mu_0 J_y$$
$$\nabla^2 A_z = -\mu_0 J_z$$

We can solve this by computing the integral

$$\mathbf{A} = \frac{\mu_0}{4\pi} \iiint \frac{\mathbf{J}(\mathbf{r}')dV'}{|\mathbf{r} - \mathbf{r}'|}$$

4 Time-varying Fields

4.1 Motional EMF

Recall that the EMF $\mathcal{E}$ is defined as the closed loop integral (usually enclosing a loop of a circuit) of force per unit charge.

$$\mathcal{E} = \oint \mathbf{f} \cdot d\ell$$

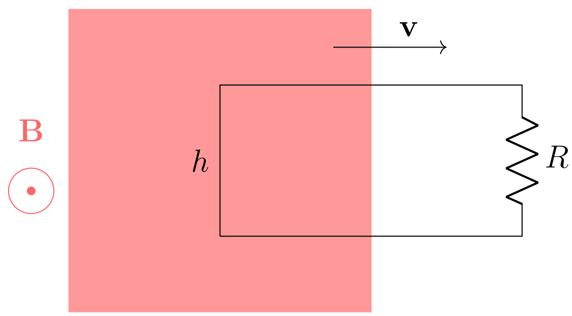
With just a battery (or another source of EMF, like a Van de Graaff generator) and current carrying wire, this reduces to the voltage of the battery because the force per unit charge can be decomposed into $\mathbf{f} = \mathbf{E} + \mathbf{f}_s$, and the closed loop integral of a static $\mathbf{E}$ is 0 by definition.

However, this changes in the presence of a magnetic field. Recall that the force on a charge moving in a magnetic field with some velocity $\mathbf{v}$ is

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

Because the force is dependent on some net motion of the particles, any effect a magnetic field has is dependent on the relative motion/change between it and the "entity" (usually a circuit) it is affecting.

For example, take a loop of wire with a resistor moving with a speed **v** relative to a region with a constant magnetic field pointing out of the page **B**.



The EMF generated by this movement can be found via

$$\mathcal{E} = \oint \mathbf{v} \times \mathbf{B} \cdot d\ell = \oint (vB)(-\hat{k}) \cdot d\ell$$

However, $\mathbf{v} \times \mathbf{B}$ points vertically, down the page. Therefore, only the components of the loop that are vertical (and are also affected by the magnetic field) factor into this line integral.

$$\mathcal{E} = -vBh$$

and the subsequent induced current in the loop is $\frac{vBh}{R}$, counterclockwise.

More generally, a change in *magnetic flux* in a loop of wire generates an EMF.

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt} \iint \mathbf{B} \cdot d\mathbf{A}$$

The negative sign in front of the time derivative accounts for Lenz's law. This is known as the flux rule.

4.1.1 Lenz's Law

Lenz's law states that the direction of an induced current in a conducting wire opposes a change in magnetic flux enclosed by the wire (As Prof. Lang puts it, "nature abhors a change in magnetic flux"). In the example above, as the loop moves out of the region with a magnetic field, it is *losing magnetic flux* pointing out of the page; to counteract this, the current induced will try to "add more dots," so to speak, and contribute towards the outwards magnetic field to make up for the loss in flux. By the right hand rule, we know that this corresponds to a current flowing counterclockwise in the loop.

If the field in the diagram above had instead been pointing into the page, we would have had a current flowing *clockwise*, to make up for the loss in inwards flux as the loop moves out of the magnetic field region.

Had the loop been moving into the magnetic field region, we would have seen the opposite effect; it would have attempted to counteract the addition of more "dots," or more outwards flux, and a current would have flowed in the clockwise direction to induce a magnetic field that points inwards. A symmetric conclusion can be drawn in the case that $\mathbf{B}$ points inwards in the first place.

4.1.2 Faraday's Law

It is often the case that the some sort of force is generated that causes a current to flow, even though the wire in question is not moving. It is thus not entirely accurate to attribute this to the induced $\mathcal{E}$ — instead, a more complete picture comes from thinking about how changing magnetic fields induce *electric fields*. This is formalized in Faraday's Law:

$$\oint \mathbf{E} \cdot d\ell = -\frac{d}{dt} \iint \mathbf{B} \cdot d\mathbf{A}$$

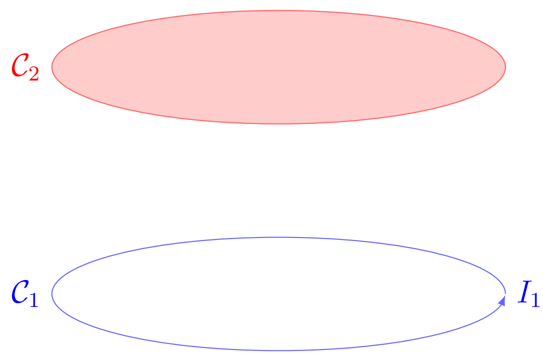
The differential version of this law can be found through Stokes' theorem, and says that

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

This is a revision to the law of electrostatics that stated that $\mathbf{E}$ was a *conservative* and *curl-free* field. As we now see, that only holds in special cases, or electrostatic conditions; Faraday's law captures the full picture with regards to this specific interaction.

4.2 Mutual Inductance

Two current loops in proximity to each other can induce effects on the other. Consider two loops of wire $\mathcal{C}_1, \mathcal{C}_2$, one of which has some current I_1 that flows through it.



The current I_1 in $\mathcal{C}_1$ induces a magnetic field that pierces the area enclosed by $\mathcal{C}_2$, thus creating magnetic flux through $\mathcal{C}_2$ which is directly proportional to I_1. We define a quantity called the *mutual inductance* from $\mathcal{C}_1$ to $\mathcal{C}_2$:

$$M_{12} = \frac{\Phi_{12}}{I_1}$$

where Φ_{12} refers to the flux in loop 2 from the field induced by loop 1. M_{12} is essentially the flux in the loop normalized for the current that induces it.

If I_1 varies with time, then the EMF induced in $\mathcal{C}_2$ is

$$\mathcal{E}_2 = -\frac{d\Phi_{12}}{dt} = -M_{12}\frac{dI_1}{dt}$$

which is, on the whole, much easier to compute if M_{12} and I_1 is known than finding the flux and then taking its derivative.

This quantity M is measured in henries (H).

4.2.1 Reciprocity Theorem

It turns out that $M_{12} = M_{21} = M$ — that is, it does not matter which curve is receiving the flux and which curve is inducing it. This means that the mutual inductance M of a system of loops is a property of the geometry of the system itself — it is almost analogous to capacitance for a system of two sheets.

$$\frac{\Phi_{12}}{I_1} = \frac{\Phi_{21}}{I_2} = M$$

The Neumann formula gives a strictly geometrical method of finding M:

$$M = \frac{\mu_0}{4\pi}\oint_{\mathcal{C}_1}\oint_{\mathcal{C}_2}\frac{d\ell_1 \cdot d\ell_2}{r}$$

4.3 Inductors

Consider a conducting loop with a variable current flowing through it. As the current in the loop changes, the magnetic flux through its own interior changes as well, which induces an EMF. We define a "self-inductance" L to be

$$L = \frac{\Phi}{I}$$

where I is the current that runs through the loop and Φ is the subsequent flux. This is also measured in henries. The EMF induced by the variable current will therefore be

$$\mathcal{E} = -L\frac{dI}{dt}$$

that is, inductors resist a change in current. This EMF is sometimes called the "back-EMF" for this reason.

Inductors introduce an inertial effect into circuits, in that it naturally resists changes in current, analogous to how mass resists changes in motion in mechanical systems.

The total energy that can be stored in an inductor is

$$U = \frac{1}{2}LI^2$$

as it takes work to "charge up" an inductor. This also lends itself to the notion of energy (or, energy density) stored in a *magnetic field*:

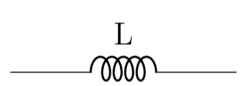
$$u_B = \frac{1}{2\mu_0}B^2$$

$$U_B = \iiint \frac{1}{2\mu_0}B^2 dV$$

Compare this to the energy density of the electric field:

$$u_E = \frac{\epsilon_0}{2}E^2$$

In circuits, inductors are represented as coils (due to their common presentation as solenoids).



The traditional KVL (the voltage drops around a closed loop equal 0) no longer holds due to magnetic effects. Instead, we have Faraday's law:

$$\oint \mathbf{E} \cdot d\ell = -\frac{d\Phi_B}{dt} = -L\frac{dI}{dt}$$

If put in series with something like a resistor, we get an equation that looks like

$$-\mathcal{E}_{battery} + IR = -L\frac{dI}{dt}$$

If we move the inductance term to the other side, we get something that looks *like* Kirchhoff's loop rule once again:

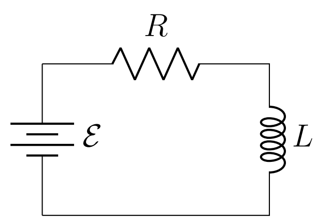
$$\mathcal{E}_{battery} - IR - L\frac{dI}{dt} = 0$$

and the quantity $L\ dI/dt$ mathematically functionally acts as the inductor's "voltage drop," though this is physically inaccurate. Indeed, $L\ dI/dt$ is what a voltmeter would read if placed in parallel with an inductor.

When current is *increasing* through an inductor, the back-EMF will be positive, and this will take away from the overall EMF of the circuit, as the inductor fights back against the influx of current. This corresponds to a voltage "gain." When current is decreasing, the quantity $L\ dI/dt$ will be negative, and so the inductor will *help the current through* and contribute a positive quantity to the overall EMF of the circuit. This corresponds to a voltage "drop."

4.4 Circuits

4.4.1 LR



From integrating around the loop, we get the equation

$$\mathcal{E} - IR - L\frac{dI}{dt} = 0$$

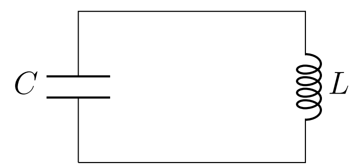
Solving this yields

$$I(t) = \frac{\mathcal{E}}{R}(1 - e^{-Rt/L})$$

Analogous to an RC circuit, the time constant of this circuit is

$$\tau = \frac{L}{R}$$

4.4.2 LC



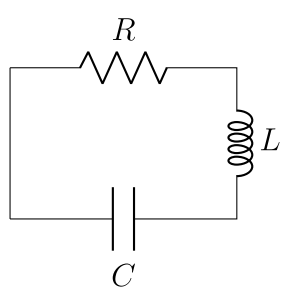
In this case, either the capacitor or the inductor must have stored some initial energy to "discharge" into the other, acting as an initial power source.

$$\frac{Q}{C} - L\frac{dI}{dt} = 0$$
$$\frac{d^2Q}{dt^2} = \frac{1}{LC}Q(t)$$

This is reminisicent of a simple harmonic oscillator. Assuming perfect capacitors and inductors, there will be no energy dissipated from the circuit; the charge will constantly oscillate back and forth with an angular frequency of

$$\omega = \frac{1}{\sqrt{LC}}$$

4.4.3 RLC



Extending the analogy to a simple harmonic oscillator, the resistor in this circuit acts as a kind of dashpot, adding a damping factor to the differential equation.

$$\frac{Q}{C} - IR - L\frac{dI}{dt} = 0$$

The solution will take the form

$$Q(t) = Q_0 e^{-\gamma t}\cos(\omega t + \phi)$$

where

$$\gamma = \frac{R}{2L}$$

and

$$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

If you recall that $\omega_0 = \frac{1}{\sqrt{LC}}$ for an undamped LC circuit,

$$\omega = \sqrt{\omega_0^2 - \gamma^2}$$

4.5 Displacement Current

When we are no longer dealing with magnetostatic situations (i.e. when $\mathbf{J}$ is not steady), Ampere's law in its simple form no longer holds. The proof relied on a special case of the continuity equation, which, more generally, is

$$\oiint \mathbf{J}\cdot d\mathbf{A} = -\frac{dQ_{enc}}{dt} \Longleftrightarrow \nabla\cdot\mathbf{J} = -\frac{d\rho}{dt}$$

Instead, we require a correction factor that is called Maxwell's displacement current.

$$I_{disp} = \epsilon_0 \frac{d}{dt}\iint \mathbf{E}\cdot d\mathbf{A}$$

$$\mathbf{J}_{disp} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

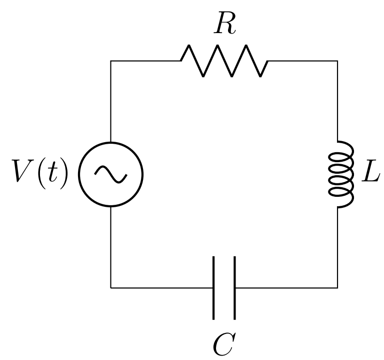
The more complete version, the Ampere-Maxwell law, is formalized as:

$$\begin{aligned}
\oint \mathbf{B} \cdot d\ell &= \mu_0 (I_{enc} + I_{disp}) \\
&= \mu_0 I_{enc} + \mu_0 \epsilon_0 \frac{d}{dt} \iint \mathbf{E} \cdot d\mathbf{A} \\
\nabla \times \mathbf{B} &= \mu_0 (\mathbf{J} + \mathbf{J}_{disp}) \\
&= \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}
\end{aligned}$$

5 AC Circuits

AC circuits are driven by a variable, oscillating source of EMF (often in the form of some sinusoid)

$$V(t) = V_0 \cos(\omega t)$$



The solution $I(t)$ will be of the form of some

$$I(t) = I_0 \cos(\omega t - \phi)$$

where I_0 is the amplitude of the current and ϕ is the phase lag.

Often, people care about the root mean squared voltage:

$$V_{RMS} = \sqrt{\langle (V(t))^2 \rangle} = \sqrt{\langle V_0^2 \cos^2 \omega t \rangle}$$

The angle brackets indicate the time average of some function, often over its period. For a periodic function $f(t)$ with a period T,

$$\langle f(t) \rangle = \frac{1}{T} \int_0^T f(t) dt$$

The period averages of $\cos^2$ and $\sin^2$ end up being $\frac{1}{2}$. Therefore,

$$V_{RMS} = V_0\sqrt{\langle\cos^2\omega t\rangle} = \frac{V_0}{\sqrt{2}}$$

5.1 Complex Representations

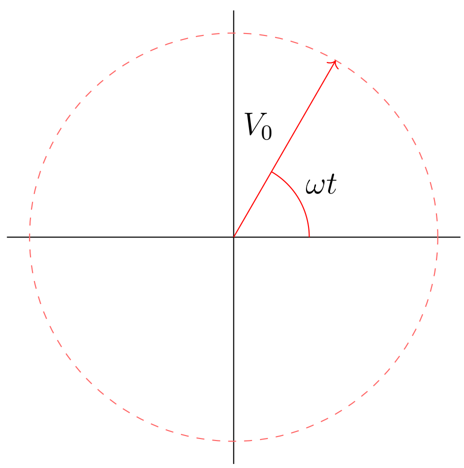
Instead of carrying sinusoidal functions around to solve for $I(t)$, we can instead express the varying EMF as a complex exponential:

$$\tilde{V} = V_0 e^{i\omega t}$$

such that

$$V(t) = Re\left[\tilde{V}\right]$$

This corresponds to a complex representation known as a phasor.



At any given time, the projection of this phasor vector onto the real axis gives the actual physical voltage being driven into the circuit.

We can then complexify the quantities we are trying to solve for and take the real part of the final solution to find our desired current/charge. This is because derivatives, integrals, and scalar multiplications are linear operations, so they affect complex numbers *piecewise* — the real part won't affect the complex part, and vice versa.

$$\tilde{V} - R\tilde{I} - L\frac{d\tilde{I}}{dt} - \frac{\tilde{Q}}{C} = 0$$

We know that our real solution $I(t)$ should be of some form

$$I(t) = I_0 \cos(\omega t - \phi)$$

for an amplitude I_0 and a phase lag ϕ. In complex form,

$$\tilde{I(t)} = I_0 e^{-i\phi} e^{i\omega t} = \tilde{I}_0 e^{i\omega t}$$

so the information about the real current is encoded in the complex amplitude $\tilde{I}_0$. Plugging this in,

$$\tilde{I}_0 e^{i\omega t} R + Li\omega \tilde{I}_0 e^{i\omega t} + \frac{1}{i\omega C} \tilde{I}_0 e^{i\omega t} = V_0 e^{i\omega t}$$

Simplification yields

$$\tilde{I}_0 R + Li\omega \tilde{I}_0 + \frac{1}{i\omega C} \tilde{I}_0 = V_0$$

which we can solve for $\tilde{I}_0$. Once we have done this, we can directly read off I_0 and ϕ from knowing that

$$\tilde{I}_0 = I_0 e^{-i\phi}$$

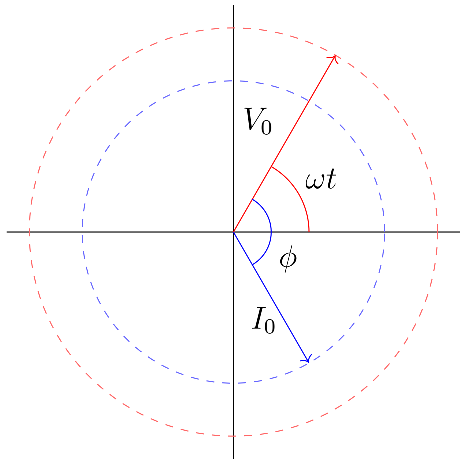
In the series RLC circuit, these quantities turn out to be

$$I_0 = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

$$\tan\phi = \frac{\omega L}{R} - \frac{1}{\omega R C}$$

If ϕ is positive, the current *lags* the voltage; if ϕ is negative, the current *leads* the voltage. A current through an inductor on its own will lag the voltage (as it resists a change in current), while a current through a capacitor on its own will lead the voltage (as it resists a change in voltage). The phase lag or lead of the two in series will depend on the individual strengths of the components.

On the phasor diagram, this will look something like



This series RLC circuit reaches *resonance*, i.e. the maximum amplitude of current, when

$$\omega = \frac{1}{\sqrt{LC}}$$

At resonance, there will be no phase lag, and the current will be *in phase* with the voltage.

5.2 Impedance

A much cleaner way of solving RLC circuits is via conceptualizing *complex impedance*, which is defined to be

$$Z = \frac{\tilde{V}(t)}{\tilde{I}(t)}$$

for a given component. For an inductor,

$$Z_L = i\omega L$$

and for a capacitor,

$$Z_C = \frac{1}{i\omega C}$$

For a resistor, since all of its quantities are real,

$$Z_R = R$$

Mathematically, impedances act as "complex resistances"; they add in series and in parallel exactly as resistors do:

$$Z_{eq,||} = Z_1 + Z_2 + \cdots + Z_n$$
$$\frac{1}{Z_{eq,\perp}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_n}$$

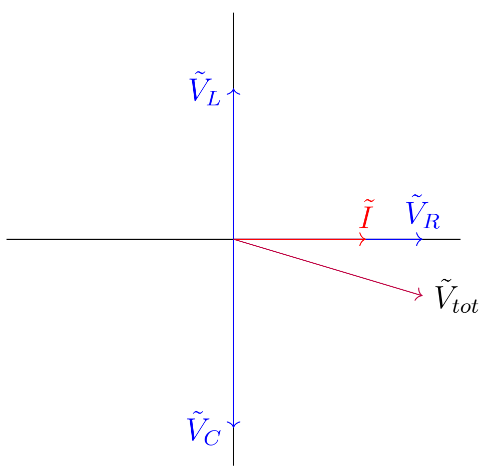
Impedances also generalize Ohm's law; that is,

$$\tilde{V} = \tilde{I}Z$$

We can get the current easily by computing the equivalent impedance of the circuit and taking the real part.

$$I(t) = Re\left[\frac{V_0 e^{i\omega t}}{Z_{eq}}\right]$$

On a phasor diagram, the impedances of inductors and capacitors are "vectors" in the imaginary basis, so they add to comprise the imaginary component of the equivalent impedance (which also corresponds to the imaginary components of the complex voltage and current). Any impedance contributed by the resistor remains the basis of the real part of these quantities.



5.3 Power

The source of an AC circuit provides power

$$P(t) = I(t)V(t)$$

Note that I and V are *real* quantities; we must take the real component of the current and the voltage before computing their product, as the product is not necessarily a linear operation.

$$P(t) = Re\,[I(t)]\,Re\,[V(t)]$$

Taking the time average of this over a period ends up yielding

$$\langle P(t)\rangle = \frac{V_0^2}{2|Z_{eq}|}\cos\phi$$

which ends up being a function of ω.

$$\langle P(\omega)\rangle = \frac{V_0^2}{2|Z_{eq}(\omega)|}\cos\phi(\omega)$$

$\cos\phi$ is known as the *power factor*.

$$\cos\phi = \frac{Re[Z_{eq}]}{|Z_{eq}|}$$

We can also write this power supplied in terms of root-mean-squared quantities:

$$I_{RMS} = \frac{V_{RMS}}{|Z_{eq}|} = \frac{I_0}{\sqrt{2}}$$

and

$$\langle P(\omega)\rangle = \frac{V_{RMS}^2}{|Z_{tot}|}\cos\phi$$

6 Electromagnetic Waves

In the 1860s, Maxwell showed that the solutions to the set of four equations describing electromagnetic interactions were waves.

Waves are functions $\psi(x,t)$ that satisfy the wave equation, a partial differential equation of the form

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

where v is the speed of the wave.

$$v = \frac{\Delta x}{\Delta t}$$

It turns out that the input to ψ must be of the form $x-vt$ for a wave moving to the right (or $x+vt$ for a wave moving to the left).

Suppose we have two fields of the form

$$\mathbf{E}(\mathbf{r},t) = E_y(x,t)\hat{j}$$
$$\mathbf{B}(\mathbf{r},t) = B_z(x,t)\hat{k}$$

If we utilize Faraday's law and Ampere's law and combine their equations together with the wave equation, we find that

$$\frac{\partial^2 E_y}{\partial x^2} = \mu_0\epsilon_0\frac{\partial^2 E_y}{\partial t^2}$$
$$\frac{\partial^2 B_z}{\partial x^2} = \mu_0\epsilon_0\frac{\partial^2 B_z}{\partial t^2}$$

which indicates that

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$$

and that electromagnetic waves are the propagators of light.

In vacuum, the most basic form of a wave solution looks like

$$\mathbf{E} = E_0 \sin\left(\frac{2\pi}{\lambda}(x-ct)\right)\hat{y}$$
$$\mathbf{B} = B_0 \sin\left(\frac{2\pi}{\lambda}(x-ct)\right)\hat{z}$$

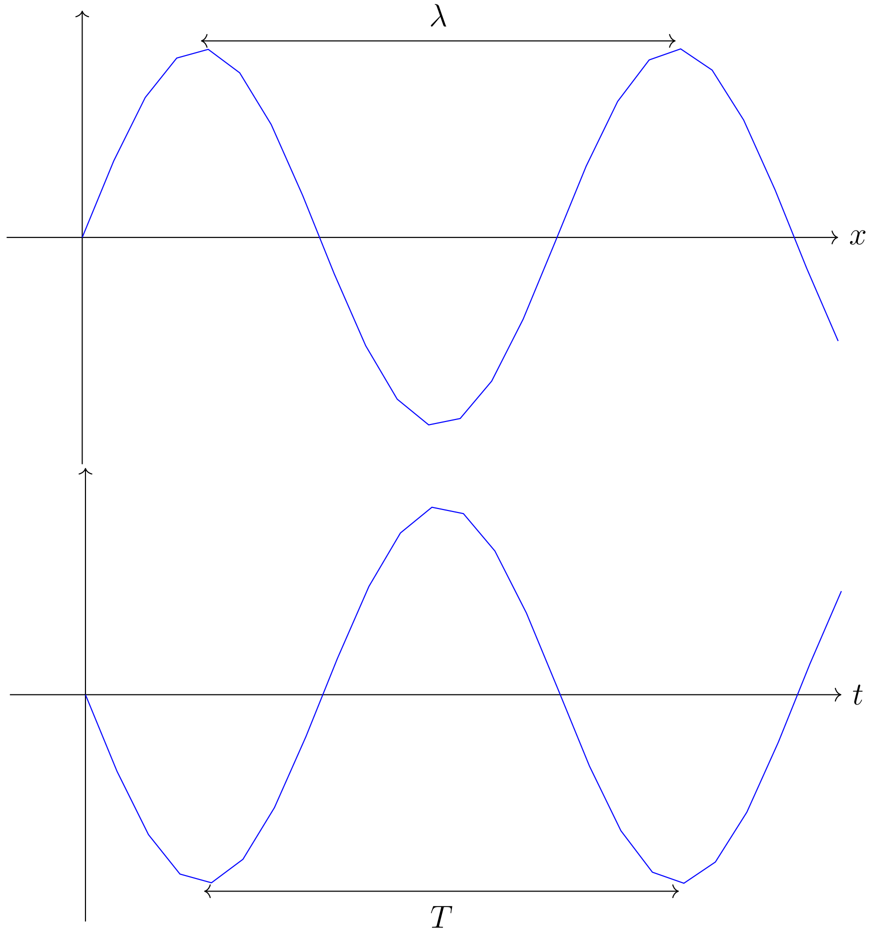
We could have chosen any unit direction for $\mathbf{E}$, and any arbitrary phase effect (including changing sin to cos). It will satisfy the equations either way. In the cases above, the waves are propagating in the $\hat{x}$ direction.

Notice that we have chosen our $\mathbf{E}$ and $\mathbf{B}$ to be paired in some way. In fact, for a generic electric field polarized in some unit direction (in this case, $\hat{y}$), and with a given phase angle, $\mathbf{B}$ must point in a direction orthogonal to it and the direction in which $\mathbf{E}$ is polarized in a right-handed way. In fact, the direction of propagation of a wave $\hat{k}$ (distinct from the convention that $\hat{k}$ is used to refer to the z-direction) must point in the direction of $\mathbf{E}\times\mathbf{B}$. In addition, the phase effects and wave qualities (frequency, wavelength, etc) must correspond. Lastly,

$$B_0 = \frac{E_0}{c}$$

6.1 Wave Properties

The spatial component of the wave and the time-dependent component of the wave can be plotted separately.



λ, or the wavelength, is a "spatial period"; there is one oscillation every wavelength λ. T is the "temporal period"; there is one oscillation every T.

$$T = \frac{\lambda}{c}$$

We can define the wavenumber

$$k = \frac{2\pi}{\lambda}$$

and the angular frequency

$$\omega = \frac{2\pi c}{\lambda}$$

The frequency of the wave is defined as

$$f = \frac{1}{T} = \frac{c}{\lambda}$$

In addition,

$$\frac{\lambda}{T} = \lambda f = c$$
$$\frac{\omega}{k} = c$$

so components of waves can be expressed as

$$E_0 \sin(kx - \omega t)$$

In general, when the direction of propagation of a wave is not expressable with just the elementary unit vectors, k becomes a vector quantity $\mathbf{k}$ with magnitude k and direction $\hat{k}$, where $\hat{k}$ points in the direction of propagation. A more complete set of equations is

$$\mathbf{E} = E_0 \sin(\mathbf{k} \cdot \mathbf{r} - \omega t - \phi)\hat{e}$$
$$\mathbf{B} = \frac{E_0}{c} \sin(\mathbf{k} \cdot \mathbf{r} - \omega t - \phi)\hat{b}$$

where $\hat{e}$ and $\hat{b}$ must be orthogonal to each other and

$$\hat{e} \times \hat{b} = \hat{k}$$

6.2 Energy

Recall the energy densities of the electric and magnetic fields:

$$u_E = \frac{\epsilon_0}{2} E^2$$
$$u_B = \frac{1}{2\mu_0} B^2$$

The total energy density at a point for an electromagnetic wave is the sum of these two components.

$$u = u_E + u_B = \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2$$

and therefore

$$dU = udV$$

A piece of volume traversed by a wave piercing some area A creates a volume element

$$dV = Acdt$$

so

$$dU = uAcdt$$

The rate at which energy flows per area is, therefore,

$$\frac{dU}{dt} = cu = c\left(\frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2\right)$$

Using the relation that $B = \frac{E}{c}$, we can derive that

$$\frac{dU}{dt} = \frac{1}{\mu_0}EB$$

This is defined to be the quantity S.

6.2.1 Poynting Vector

The Poynting vector imbues a direction onto S; namely, the direction of the propagation of the wave.

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$$

The intensity of a wave is the time-averaged magnitude of the Poynting vector:

$$I = \langle|\mathbf{S}|\rangle$$

Since the poynting vector has squared sinusoids, the time-average ends up adding a factor of $\frac{1}{2}$.

For a plane wave,

$$I = \frac{1}{2}c\epsilon_0 E_0^2 = c\langle u\rangle$$

For a spherical wave, the distance at a distance r from the center is

$$I(r) = \frac{L}{4\pi r^2}$$

where L is the luminosity of the light source.

Light hitting a surface exert some sort of radiation pressure onto the surface. For a perfectly absorbing surface,

$$P_{rad} = \frac{I}{c}$$

For a perfectly reflecting surface,

$$P_{rad} = \frac{2I}{c}$$

The Poynting vector can more generally describe energy flow in any electromagnetic situations, not just waves. We have the following law:

$$\oiint \mathbf{S} \cdot d\mathbf{A} = -\frac{dU}{dt}$$
$$\nabla \cdot \mathbf{S} = -\frac{\partial u}{\partial t}$$

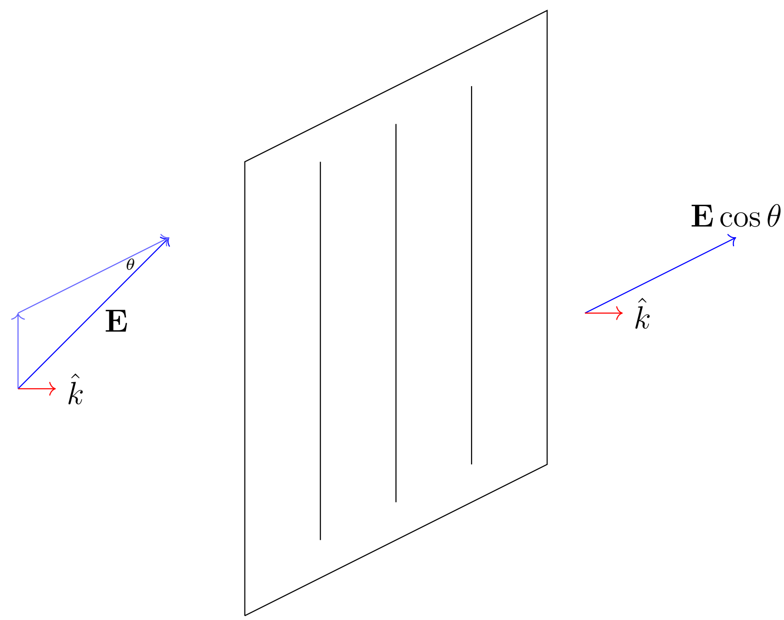
This is a conservation law, just like the continuity equation for current density and charge.

6.3 Polarization

The polarization of a wave is defined as the direction in which the electric field of the wave points. A polarizer is a type of material that only lets light of a certain polarization, or direction, through; more formally, a factor of $\cos\theta$, the angle to the direction perpendicular to the polarizing chains, through. The ratio of the final observed intensity and the initial intensity of the light is

$$\frac{I_f}{I_0} = \cos^2\theta$$

This is known as Malus' law.



6.4 Generalizations

The wave equation works in all three dimensions, not just along one axis. In general:

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$
$$\nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

Or, in general,

$$\nabla^2 \psi = \mu_0 \epsilon_0 \frac{\partial^2 \psi}{\partial t^2}$$

for any solution ψ.

We define a new operator, the *D'Alembertian*:

$$\Box = \nabla^2 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2}$$

such that

$$\Box \psi = \nabla^2 \psi - \mu_0 \epsilon_0 \frac{\partial^2 \psi}{\partial t^2}$$

which equals 0 in vacuum.

What if we aren't in vacuum, and we have sources for charges and currents (which therefore generate electric and magnetic fields)? We have the *inhomogeneous wave equations*, which relate these sources to *potentials*:

$$\Box V = -\frac{\rho}{\epsilon_0}$$
$$\Box \mathbf{A} = -\mu_0 \mathbf{J}$$

As it turns out, equations for magnetostatic and electrostatic situations (e.g. Poisson's equation) describe the *steady state* of the system as a test charge receives information from its surroundings, limited by the speed of light. With the D'Alembertian, we have a more complete picture of what happens between a source being turned on and when the steady state is reached.

7 Special Relativity

7.1 Postulates

The postulates of special relativity are as follows:

1. Principle of Relativity: the laws of physics (e.g. Newton's laws, conservation of energy, Gauss' law) are the same in all inertial reference frames.

2. The speed of light c is the same in all reference frames. This implies that light propagates without a specific medium.

An inertial reference frame is one in which Newton's first law holds — an object feeling no net force will travel at constant velocity. Any reference frame moving at a constant velocity v to an inertial reference frame will also be inertial.

A reference frame S depends on four coordinates: (x, y, z), the spatial coordinates, and t, time. An event (anything happening, like a light flash) is recorded and is described by its set of coordinates (x, y, z, t). The objective is to see how these coordinates transform when one event is looked at from different inertial reference frames, each moving at a constant velocity from each other.

7.2 Lorentz Transformations

At relativistic speeds, classical Galilean transforms between coordinate frames no longer hold. Instead, given a reference frame $S(x, y, z, t)$ and another reference frame $S'(x', y', z', t')$ moving with a relative velocity v to S along the x axis, some event (x, y, z, t) will be transformed in the following way:

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{v}{c^2}x\right) \end{aligned}$$

Note that y and z, coordinates not along the axis to which the frame S' is moving, stay the same across both systems. The inverse transformations look similar, and can be thought of as reframing the situation such that S' is stationary and S is moving at a velocity $-v$ to S' instead:

$$\begin{aligned} x &= \gamma(x' + vt') \\ y &= y' \\ z &= z' \\ t &= \gamma\left(t' + \frac{v}{c^2}x'\right) \end{aligned}$$

where γ is the Lorentz factor:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The quantity v/c is sometimes referred to as β for the sake of cleaning up notation, so γ can also be expressed as

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}$$

and

$$\begin{aligned} x' &= \gamma(x - \beta ct) \\ ct' &= \gamma\left(ct - \beta x\right) \end{aligned}$$

In special relativity, lengths contract (get shorter) and time dilates (feels longer). For an object of length L' at rest in S', a reference frame moving with respect to the frame S, then we will measure a length L to be

$$L = \frac{L}{\gamma}$$

Similarly, for a time interval $\Delta t'$ measured in S', the interval will be measured as

$$\Delta t = \gamma \Delta t'$$

in S.

7.2.1 Spacetime Interval

Define Δs to be the following expression:

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

or,

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta r)^2$$

where the Δs refer to the fact that we are measuring "distance" between two events.

Δs is an invariant across the Lorentz transformation; it can be shown that

$$(\Delta s)^2 = (\Delta s')^2$$

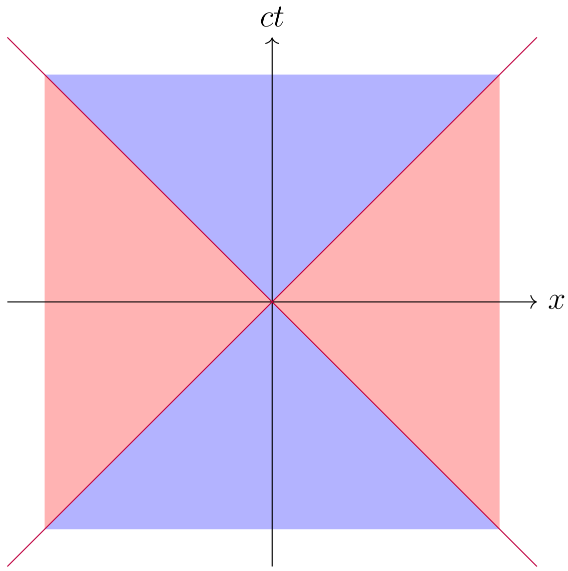
after $\Delta s'$ is computed with transformed coordinates (x', y', z', t'). It is of note that this invariant is similar to the norm invariant under length-preserving transformations in Euclidean spaces (i.e. rotations and reflections, those of determinant ± 1).

If $(\Delta s)^2 > 0$, the events are called *timelike*; this indicates that there is enough time for two events in spacetime to be connected via a traveling signal (as signals cannot travel faster than c), and therefore can be connected via conceptions of causality. In this case, there exists a reference frame in which the two events occur in the same place (just at different times).

If $(\Delta s)^2 < 0$, the events are called *spacelike*; there is no reference frame in which both events could have occurred at the same place, though there exists a reference frame in which both events could have occurred at the same time. No causal correlation can be inferred for spacelike events.

If $(\Delta s)^2 = 0$, the events are called *lightlike*; this represents the boundary between spacelike and timelike, and usually demarcates the events that can occur/communicate via signals transmitted by light.

This interval demarcates *hyperbolic* symmetries between events. We can delineate a *light cone* (the blue region in the diagram); all timelike events happen within the light cone. All spacelike events happen outside of the light cone. All lightlike events happen along the purple lines, where $x = ct$.



In generality, this interval is described as a *Minkowski metric*.

7.3 Velocity Transformations

Consider an object moving in a frame s with some constant velocity $\mathbf{u} = u_x\hat{i} + u_y\hat{j} + u_z\hat{k}$. In a frame S' moving with velocity v relative to S,

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}$$
$$u'_y = \frac{u_y}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}$$
$$u'_z = \frac{u_z}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}$$

The inverse transformations look like

$$u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}}$$

$$u_y = \frac{u'_y}{\gamma\left(1 + \frac{u'_x v}{c^2}\right)}$$

$$u_z = \frac{u'_z}{\gamma\left(1 + \frac{u'_x v}{c^2}\right)}$$

7.4 Energy and Momentum

A particle of mass m moving at velocity $\mathbf{u}$ in some frame, its relativistic momentum is

$$\mathbf{p} = \gamma_u m \mathbf{u}$$

where

$$\gamma_u = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Note that if u is tiny, the γ_u factor approaches 1, and we find our formula for momentum from classical mechanics.

The energy of this particle is

$$E = \gamma_u m c^2$$

In some frame S',

$$p'_x = \gamma\left(p_x - \frac{v}{c^2}E\right)$$

and

$$E' = \gamma(E - v p_x)$$

The other components of momentum do not change.

7.5 Four-vectors

The transformation equations for velocity/time and momentum/energy look pretty similar; in fact, they fall under the same class of object, known as four-vectors.

Define the transformation Λ as

$$\Lambda = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We note that

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \Lambda \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} E'/c \\ p'_x \\ p'_y \\ p'_z \end{bmatrix} = \Lambda \begin{bmatrix} E/c \\ p_x \\ p_y \\ p_z \end{bmatrix}$$

Any set of coordinates whose transformations between frames can be described by Λ is a four-vector. In addition, any four-vector x must satisfy the additional Minkowski metric (spacetime interval); that is, it must fulfill

$$x^T \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x = (x')^T \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x'$$

7.6 Force Transformations

The force **F** on a particle moving with some velocity **u** in a rest frame will look differently from the perspective of a moving frame.

$$F'_x = \frac{F_x - \frac{v}{c^2}\mathbf{F} \cdot \mathbf{u}}{1 - \frac{u_x v}{c}^2}$$

$$F'_y = \frac{F_y}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}$$

$$F'_z = \frac{F_z}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}$$

7.7 Field Transformations

Electric fields moving in S' with speed v are also transformed. Crucially, only components of the field *orthogonal* to the direction of motion are changed; the parallel components stay the same.

$$\mathbf{E}'_\perp = \gamma \mathbf{E}_\perp$$
$$\mathbf{E}'_{||} = \mathbf{E}_{||}$$

The electric field of a point charge moving with velocity $\mathbf{v}$ is expressed as

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{(1 - v^2 \sin^2\theta/c^2)^{3/2}} \frac{\hat{r}}{r^2}$$

Moving charges generate magnetic fields, and magnetic fields in turn influence the transformations on electric fields. More generally, field transformations follow

$$\begin{aligned}
E'_x &= E_x \\
E'_y &= \gamma(E_y - vB_z) \\
E'_z &= \gamma(E_z + vB_y) \\
B'_x &= B_x \\
B'_y &= \gamma\left(B_y + \frac{v}{c^2}E_z\right) \\
B'_z &= \gamma\left(B_z - \frac{v}{c^2}E_y\right)
\end{aligned}$$

We can conceptualize a six-component "unified electromagnetic field" from these transformations. We define the electromagnetic field tensor $F^{\mu\nu}$, a 4×4 antisymmetric matrix:

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix}$$

such that

$$(F^{\mu\nu})' = \Lambda F^{\mu\nu} \Lambda$$

7.8 Doppler Shift

The eigenvectors of the Lorentz transformation matrix Λ are

$$\lambda = 1,\ -1,\ \sqrt{\frac{1+\beta}{1-\beta}},\ \sqrt{\frac{1-\beta}{1+\beta}}$$

The non-unity eigenvalues correspond to the *doppler shift* of a light wave observed between an observer at rest and an emitter moving at relativistic speeds. Its eigenvectors are the lightlike worldlines ($x = ct$) on a Minkowski diagram.

8 Maxwell's Equations

8.1 Integral Form

$$\oiint \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{enc}}{\epsilon_0}$$
$$\oiint \mathbf{B} \cdot d\mathbf{A} = 0$$
$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \iint \mathbf{B} \cdot d\mathbf{A}$$
$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc} + \mu_0 \epsilon_0 \frac{d}{dt} \iint \mathbf{E} \cdot d\mathbf{A}$$

8.2 Differential Form

$$\nabla \cdot \mathbf{E} = \frac{\rho_0}{\epsilon_0}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$